PROCEEDINGS BOOK of MICOPAM 2023

The 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas

Dedicated to Professor Yilmaz SIMSEK

on the Occasion of his 60th Anniversary

Université d'Evry Paris-Saclay

Paris, FRANCE
August 23–27, 2023

Editors

Abdelmejid BAYAD
Mustafa ALKAN
Irem KUCUKOGLU
Ortaç ÖNEŞ

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Conference Venue
Université d'Evry Paris-Saclay

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TITLE

Proceedings Book of the 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023)

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ON THE OCCASION OF HIS 60TH ANNIVERSARY

CONFERENCE VENUE

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ABOUT CONFERENCE

The MICOPAM conference series has been started by organizing it in Antalya-Turkey in 2018, and the latter has been held in Paris, France in 2019. Then, the third one, which was postponed in 2020 due to the coronavirus pandemic, was held in Antalya, Turkey as MICOPAM 2020-2021 together with the fourth one. The fifth one has also been held in Antalya, Turkey in 2022. Over the last five years, this conference series has brought together the researchers, who work on pure & applied mathematics and related areas, from all over the world.

As for the 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023), under the organization of Professor Abdelmejid Bayad, it was held at Université d’Evry in Paris, FRANCE, for five days from August 23 to August 27, 2023, by dedicating this wonderful conference to the respected mathematician Professor Dr. Yilmaz Simsek on the occasion of his 60th anniversary.

The aim of the conference MICOPAM 2023 was to bring together leading scientists of the pure and applied mathematics and related areas to present their research, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other.

MICOPAM 2023 conference took place in a hybrid form with both Physical and Virtual (Online) participations, for five days from August 23 to August 27, 2023. There were totally 100 participants from 22 different countries. One of these participants attended the conference as a listener, and 3 participants attended the conference with a joint paper accompanied by one of the other participants. In addition, there were a total of 4 participants who made two presentations among others.

During the five days of MICOPAM 2023, its participants made totally 100 oral and poster presentations, and these contributions are respectively affiliated with 22 different countries [Algeria (6), Canada (1), Croatia (3), China (3), France (6), Germany (1), India (6), Iraq (1), Japan (1), Kuwait (1), Northern Cyprus (4), Poland (1), Portugal (1), Romania (1), Russia (4), Saudi Arabia (2), Serbia (2), South Korea (8), Spain (1), Turkey (44), United Kingdom (1), USA (2)].

In addition to a great number of excellent presentations, there was one listener participant who have supported our conference by their presence.

MICOPAM 2023 conference welcomed speakers whose talk or poster contents are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure & Applied Mathematics and Related Areas, Mathematical Statistics and its applications, Recent Advances in General Inequalities, Mathematical Physics, Fractional Calculus and its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, q-analysis and its applications, Approximation Theory and Optimization, Extremal Problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Differential Equations and their applications, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and Computation in Engineering, Mathematical Geosciences, p-adic numbers, p-adic analysis and their applications, Mathematical Modelling, Marketing, Business and their applications, Mathematical Methods for Engineering Applications.

In this context, the contents of oral and poster presentations of this conference were mainly related to not only the above areas, but also their applications in various fields of mathematics and related areas.

Further details about the conference MICOPAM 2023 are given as follows:

COMMITTEES of MICOPAM 2023

Organizing Committee

International Organizing Committee

- Mustafa Alkan, (Akdeniz University, Turkey)
- Abdelmejid Bayad, (Université d'Evry, France)
- Ayse Yilmaz Ceylan, (Akdeniz University, Turkey)
- Rahime Dere, (Alanya Alaaddin Keykubat University, Turkey)
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- Mehmet Ali Özarslan, (Eastern Mediterranean University, Northern Cyprus)
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- Ezgi Polat, (Akdeniz University, Turkey)
- Buket Simsek, (Akdeniz University, Turkey)
- Fusun Yalcin, (Akdeniz University, Turkey)

Heads of the Organizing Committee

- Abdelmejid Bayad, **Committee-in-Chief**, (Université d'Evry, France)
- Mustafa Alkan, (Akdeniz University, Turkey)
- Rahime Dere, (Alanya Alaaddin Keykubat University, Turkey)
- Neslihan Kilar, (Niğde Ömer Halisdemir University, Turkey)
- Irem Kucukoglu, (Alanya Alaaddin Keykubat University, Turkey)
- Ortaç Öneş, (Akdeniz University, Turkey)

Scientific Committee

- Chandrashekar Adiga, India
- Elvan Akin, USA
- Mustafa Alkan, Turkey
- Abdelmejid Bayad, France
- Naim L. Braha, Republic of Kosova
- Isabel Cação, Portugal
- Nenad Cakić, Serbia
- Ismail Naci Cangul, Turkey
- Seçil Çeken, Turkey
- Ahmet Sinan Cevik, Turkey
- Junesang Choi, South Korea
- Fabrizio Colombo, Italy
- Marc-Antoine Coppo, France
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- Mohand Ouamar Hernane, Algeria
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- Mokhtar Kirane, United Arab Emirates
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- Dmitry Kruchinin, Russia
- Nazim I. Mahmudov, Northern Cyprus
- Branko Malešević, Serbia
- Helmuth Robert Malonek, Portugal
- Sabadini Irene Maria, Italy
- Gradimir V. Milovanović, Serbia
- Hussein Mourtada, France
- Abbas C. Movahhedi, France
- Figen Öke, Turkey
- Mehmet Ali Özarslan, Northern Cyprus
- Manuel López-Pellicer, Spain
- Tibor Poganj, Croatia
- Dora Pokaz, Croatia
- Abdalah Rababah, Jordan
- Themistocles Rassias, Greece
- Lothar Reichel, USA
- Ekrem Savas, Turkey
- Yilmaz Simsek, Turkey
- Burcin Simsek, USA
- Miodrag Spalević, Serbia
- Wolfgang Sprößig, Germany
- Hari M. Srivastava, Canada
- Marija Stanić, Serbia
- Richard Tremblay, Canada

INVITED SPEAKERS of MICOPAM 2023

(Sorted list alphabetically by speaker's last name)

- **Chandrashekar Adiga**, (University of Mysore, India)
- **Isabel Cação,** (University of Aveiro, Potugal)
- **Marc-Antoine Coppo**, (Université Nice, France)
- **Seçil Çeken**, (Trakya University, Turkey)
- **Satish Iyengar**, (University of Pittsburgh, USA)
- **Subuhi Khan**, (Aligarh Muslim University, India)
- **Taekyun Kim**, (Kwangwoon University, South Korea)
- **Daeyeoul Kim**, (Jeonbuk National University, South Korea)
- **Rolf Sören Kraußhar**, (University of Erfurt, Germany)
- **Veerabhadraiah Lokesha**, (Vijayanagara Sri Krishnadevaraya University, India)
- **Nazim I. Mahmudov**, (Eastern Mediterranean University, Northern Cyprus)
- **Gradimir V. Milovanović**, (Serbian Academy of Sciences and Arts, Serbia)
- **Hussein Mourtada**, (Université Paris, France)
- **Abbas C. Movahhedi**, (Université Limoges, France)
- **Figen Öke**, (Trakya University, Turkey)
- **Mehmet Ali Özarslan**, (Eastern Mediterranean University, Northern Cyprus)
- **Manuel López-Pellicer**, (Universitat Politécnica de Valencia, Spain)
- **Dora Pokaz**, (University of Zagreb, Croatia)
- **Burcin Simsek**, (Bristol-Myers Squibb Company, USA)
- **Marija Stanic**, (University of Kragujevac, Serbia)
- **C. K. Subbaraya**, (Adichunchanagiri University, India)

Foreword written by Professor Yilmaz Simsek

It gives me great pleasure to write this foreword for the "*6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023), dedicated to my 60th birthday*".

In my academic life, I firstly participated in the III. Turkish National Mathematics Symposium, organized by Van Yüzüncü Yıl University in 1989. After this symposium, I have participated in many scientific events such as symposiums, conferences, workshops, seminars, etc. in almost every country in the world such as in Ukrainian Mathematics Symposium, in Jangjeon Mathematical Society Conferences, in Canadian Mathematics Symposium, in USA Mathematics Symposium, in European Mathematics Symposium and conferences, and the others.

As I know that many important conferences may be named after countries, some may be dedicated to specific topics, some may be named after general mathematical names, or they may be named after mathematical societies. Some of them are dedicated to the age of researchers in mathematics (or in sciences).

I must proudly state that I have been doing research on the following topics for more than 30 years:

- Complex Analysis, Modular Forms, Elliptic Functions
- Number Theory, p-adic Numbers and their applications
- Real and Complex Analysis
- Special numbers, Special Polynomials, Generating Functions, Special Functions, Generating Functions and Special Functions Associated with Superoscillations
- Functional Analysis, Integral Equations and Transforms
- q-Series and q-Analysis
- Analytic Number Theory (Modular form and Elliptic functions, Dedekind and Hardy Sums, theta functions), Zeta type functions, Dirichlet series and Lambert-type series
- p-adic analysis and p-adic Distribution
- Umbral Algebra and Umbral Analysis
- Combinatorial Analysis and their applications
- Probability Distribution functions and their Applications
- Computational Science and Engineering, Computational Algorithms

- Algebra of the Lyndon words
- Linear Algebra and their applications
- De Bruijn sequences and application to graph theory

I am proud and honored to have the *6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023), which is dedicated to my 60th Anniversary, held at Université d'Evry in Paris, FRANCE on August 23–27, 2023*, (`https://micopam.com`). Therefore, I would like to thank the main organizer, my friend (like as my brother) and colleague Professor Abdelmejid BAYAD. He gave his big energy and time in order to prepare this wonderful and high quality international conference at Université d'Evry in Paris, FRANCE. And also, I would like to thank Professor Vincent Bouhier (President of the University of Évry Paris-Saclay) and all staff of the University of Évry Paris-Saclay for their help in order to organize this wonderful conference. In addition, I would also like to thank other organizers: my friend and colleague Professor Mustafa ALKAN, my colleague and hardworking doctoral student, Associate Professor İrem KÜÇÜKOĞLU, my colleague and hardworking doctoral student, Associate Professor Neslihan KILAR and my colleague and friend Associate Professor Ortaç ÖNEŞ, who are very valuable members of the organizing committee of this conference.

I would like to thank the other local organizers such as Assistant Professor Rahime DERE and Associate Professor Ayse Ceylan YILMAZ.

I would also like to express my sincere gratitude to the following my friends and colleagues who are invaluable and distinguished invited speakers of this conference:

- Professor Chandrashekar Adiga, (University of Mysore, India)
- Professor Isabel Cação, (University of Aveiro, Portugal)
- Professor Marc-Antoine Coppo, (Université Nice, France)
- Associate Professor Seçil Çeken, (Trakya University, Turkey)
- Professor Satish Iyengar, (University of Pittsburgh, USA)
- Professor Subuhi Khan, (Aligarh Muslim University, India)
- Professor Taekyun Kim, (Kwangwoon University, South Korea)
- Professor Daeyeoul Kim, (Jeonbuk National University, South Korea)
- Professor Rolf Sören Kraußhar, (University of Erfurt, Germany)

- Professor Veerabhadraiah Lokesha, (Vijayanagara Sri Krishnadevaraya University, India)
- Professor Nazim I. Mahmudov, (Eastern Mediterranean University, Northern Cyprus)
- Professor Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
- Professor Hussein Mourtada, (Université Paris, France)
- Professor Abbas C. Movahhedi, (Université Limoges, France)
- Professor Figen Öke, (Trakya University, Turkey)
- Professor Mehmet Ali Özarslan, (Eastern Mediterranean University, Northern Cyprus)
- Professor Manuel López-Pellicer, (Universitat Polité cnica de Valencia, Spain)
- Professor Dora Pokaz, (University of Zagreb, Croatia)
- Associate Professor Burcin Simsek, (Bristol-Myers Squibb Company, USA)
- Professor Marija Stanić, (University of Kragujevac, Serbia)
- Professor C. K. Subbaraya, (Adichunchanagiri University, India)

I sincerely thank all of my esteemed and hardworking students who honored me by attending to this conference in different cities of Turkey and other countries. Some of these students are given below:

Assistant Professor Rahmime Dere, Associate Professor İrem Kücükoğlu, Assistant Professor Elif Çetin, Associate Professor Neslihan Kilar, Associate Professor Ahmet Altürk, Assistant Professor Erkan Ağyüz, Dr. Damla Gün, Elif Şükrüoğlu, Ezgi Polat, and Elif Bozo, ...

Again, I would like to express my sincere thanks to my distinguished and valuable colleagues who participated in this conference in different countries of the world, either online or face-to-face, and made very useful and high-quality presentations at this conference.

I would like to thank my dearest wife, Saniye, who I have been married to since 1986. During my married days, she has given me most important love, effort, patience and motivation at the main source of almost all of my works and achievements. In addition, my two other favorite members of the family are my dear daughters Associate Professor Burçin Şimşek and Dr. Buket Şimşek, whom I cannot change for anything in the world. They have been the bright light of my studies and researches with their mothers, with the pleasure and motivation I felt for both my daughters and their successes.

I sincerely thank academic staff and other staff of the Université d'Evry in Paris, FRANCE for their help and support this very nice and high-quality conference.

I would also like to thank our Editors for their efforts in the preparation of the conference book with ISBN number, which is dedicated to my 60th birthday. I sincerely believe that the articles, abstracts or reviews published in this book contain results that will serve as a resource for all researchers. I would also like to thank my colleagues who supported this book by sending their articles.

Now I give briefly my mathematical philosophy:

Doing research in mathematics and related areas, writing articles, constructing a new result are the main motivations of my hobby. Thus, especially, constructing a mathematical structure and making its presentations available to researchers, thinking mathematically, mathematical creativity, and mathematical approach give me effective and decisive therapy.

When I was depressed, tired, exhausted (or because of any work), the most effective remedy I resorted to was to open my "***handbook***" and write down (a) mathematical structure(s) that come to my mind at that moment by diving very deep.

With the love of my dear family, the advice of my most respected friends and colleagues, as well as the advice of my most respected friends and colleagues, I have always written something on mathematical structures, by blending my memories with my deepest feelings, even now I write new structures, and I will write to establish new structures in the future. By sharing what I write, discover, publish, or new scientific projects with researchers in every country in the world, I increase the members of my collaborators. I am happy and honored not only to do research, but also to always train students.

Prof. Dr. Yilmaz Simsek
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Foreword written by the Editors

With our deepest respect to Professor Yilmaz Simsek, we hereby have the honor to write this foreword for the Proceedings Book of *6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas* (MICOPAM 2023), dedicated to his 60th birthday.

As we stated in the information given about the conference, the first edition of the conference series MICOPAM was held in Antalya, Turkey in 2018. Since 2018, it has been held regularly by Professor Yilmaz Simsek and his colleagues. Over the last five years, this conference series has brought together the researchers, who work on pure & applied mathematics and related areas, from all over the world.

As for this year, with the support of the University of Évry which is a member of the ComUE "Université Paris-Saclay", Professor Dr. Abdelmejid Bayad organized the conference MICOPAM 2023 in Paris, France, by dedicating the conference to the 60th birthday of Professor Yilmaz Simsek.

In addition to the its main theme, another important purpose of the MICOPAM 2023 was to bring together leading scientists of the pure & applied mathematics and related areas to present their research, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other.

As for the Proceedings Book of MICOPAM 2023, its content depends on all oral and poster presentations made in this conference and they were mainly related to pure & applied mathematics and related areas.

A brief description about the content of the Proceedings Book of MICOPAM 2023 is given as follows:

The first section of the Proceedings Book of MICOPAM 2023 includes the Welcome Speech of Professor Vincent Bouhier (President of the University of Évry Paris-Saclay), the Ceremony Talk of Professor Abdelmejid Bayad (Head of the Organizing Committee), Foreword written by Professor Yilmaz Simsek, Foreword written by the Editors, A brief biography of Professor Yilmaz Simsek, and some information about the conference series MICOPAM including the name of invited speakers, the name of committee members and the poster of the conference.

The rest of this book includes all the contributed talks and their corresponding papers.

In this regard, we would like to thank to all speakers and participants for their valuable contributions. We also express our sincere thanks to all members of the scientific committee and all members of the organizing committee because of their efforts to the success of this conference and this book.

Finally, we would like to express here that we are very happy that we celebrated the 60th birthday of very valuable and distinguished scientist Professor Yilmaz Simsek, and we also dedicate this book to his 60th birthday. For Professor Simsek's great contribution to mathematics and his guidance for young researchers, we would like to present our endless respect to him and to congratulate his 60th birthday with our best wishes once again. We wish the rest of his life to be happy, fruitful, success and he have many healthy years to spend with his loved ones.

We would also like to express our sincere appreciation to all participants who contributed to the conference MICOPAM 2023.

The Editors of the Proceedings Book of MICOPAM 2023

Prof. Dr. Abdelmejid Bayad

Prof. Dr. Mustafa Alkan

Assoc. Prof. Dr. Irem Kucukoglu

Assoc. Prof. Dr. Ortaç Öneş

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Prof. Dr. Yilmaz Simsek: A Mathematician Who Never Gives Up

I. A Brief Biography

Prof. Dr. Yilmaz Simsek received the PhD degree in (Generalized Dedekind and Hardy sums) at Cukurova University Adana of Turkey. Professor Simsek's carrier has been started from Adana, Cukurova University, Mersin University, and now continue at Antalya, Akdeniz University. His research interests are "Modular Forms, p-adic Analysis, q-series, Special numbers and polynomials, Special Functions, Generating Functions, Dedekind and Hardy Sums, Umbral Algebra, and Umbral Analysis, Special Functions, Real and Complex Analysis, etc." In addition to writing hundreds of articles and book chapters in distinguished international journals of mathematical and engineering sciences, he has succeeded in making hundreds of conferences and presenting his work to the service of many researchers in the scientific world as his academic goal and philosophy of life. He has been invited to many scientific activities such as international conferences, seminars, visiting professor. He is referee and editor of many mathematical journals. His work has received around **6800 citations** in Google Scholar with **h-index 45** for now. He has memberships of Editorial Board of many international journals. His works contributed to many areas of mathematics, especially **generating functions**. He has founded the journal "**Montes Taurus Journal of Pure and Applied Mathematics** (ISSN: 2687-4814)".

II. Other Details

Education:

- PhD. Cukurova University (Adana-Turkey) – 1993: (Generalized Dedekind and Hardy sums)
- M.Sc. Cukurova University (Adana-Turkey) – 1990: (Hermitian Matrix Inequalities)
- B.Sc. Cukurova University (Adana-Turkey) – 1987.
- Certificate/Diploma Teaching involving Courses in university pedagogy January 1988

Work Experience:

- Member of the faculty, Department of Mathematics, Faculty of Science, University of Akdeniz since 2005

- Since 2012 Professor
- 2007 – 2012 Associate Professor
- 2005 – 2007 Assistant Professor
- 1993 – 2005 Member of the faculty, Department of Mathematics, Faculty of Arts and Sciences, Mersin University
- 1993 – 2005 Assistant Professor
- 1988 – 1993 Member of the faculty, Mersin Vocational High School, Cukurova University
- 1988 – 1993 Lecturer

Awards:

- Science Award, Akdeniz University Science Antalya Turkey, Service and Encouragement Awards, November 2011
- Jangjeon Mathematical Society Award, 2017, South Korea
- Ranking of scientists in Turkish Institutions according to their Google Scholar Citations public profiles

Current Research Interests:

- Real and Complex Analysis and their applications
- Analytic Number Theory and its applications
- Modular Forms
- Elliptic Functions
- p-adic Numbers and their applications
- p-adic integrals and their applications
- p-adic analysis and p-adic Distribution
- Generating Functions and their applications
- Special numbers, Special Polynomials
- Special Functions
- Integral Equations and Transforms
- q-Series and q-Analysis

- Modular form and Elliptic functions associated with Dedekind sums and Hardy Sums, the finite sums
- Theta functions
- Families of Zeta type functions
- Dirichlet series and Lambert-type series
- Umbral Algebra and Umbral Analysis and their applications
- Combinatorial Analysis and their applications
- Probability Distribution functions and their Applications
- Computational Science and Engineering
- Computational Algorithms
- Algebra of the Lyndon words
- Linear Algebra and their applications
- De Bruijn sequences
- Generating functions and special functions associated with Superoscillations
- Applications of generating functions to areas such as the human ear and teeth,
- and so on.

International journal he founded:

- Montes Taurus Journal of Pure and Applied Mathematics ISSN: 2687-4814, `https://mtjpamjournal.com`.

Over and above, Professor Simsek continues to work as a referee and editor in many different national and international journals. Professor Simsek advises more than 30 graduate and doctoral students and continues to increase intensively. He provides trainings in many countries with the Erasmus exchange program and the Mevlana exchange program. As a Visiting Professor, Professor Simsek gives trainings and scientific studies in many countries. More than 300 articles and book chapters have been published, and he continues to work on these subjects either alone, either with students or with other international distinguished scientists. Professor Simsek has participated in more than 120 international symposiums and conferences and presented oral presentations in almost every part of the world and in most of the countries, despite the pandemic, he continues to intensify these oral presentations. In addition to these, Professor Simsek has organized many special workshops and has made very important contributions to many workshops and continues to do so. Professor Simsek has devoted his life to mathematics, and his only notable hobby is listening, studying mathematics or exploring new results. In this way, he has made it a philosophy of life that he is very peaceful and can do much more useful work for humanity. In conclusion, life is beautiful and meaningful doing and teaching math; for this purpose, it continues on its way tirelessly. With this philosophy of mathematics, Professor Simsek gave his two precious and beloved daughters, Associate Professor Dr. Burcin Simsek (Probability, Statistics and Biostatistics and their applications in medicine and other sciences) and Dr. Buket Simsek (Probability, Statistics and Biostatistics and their applications in medicine and engineering) has also chosen him as a colleague in different fields. He gained very deep knowledge from these two precious daughters. It is worth mentioning that the source of these successes is the tremendous and unforgettable support of his precious wife, Saniye Simsek. Professor Simsek sincerely believe that he will serve for many more years, and write more articles and books, thanks to the support and love of his family, whom he loves very much, the respect and support of his very valuable students, and all his distinguished friends around the world with whom Professor Simsek has collaborated and written articles. Further details about Professor Simsek's professional achievements and scholarly accomplishments, as well as honors, awards and distinctions, can be found at the following websites:

- `https://avesis.akdeniz.edu.tr/ysimsek`,
- `https://orcid.org/0000-0002-0611-7141`
- `https://scholar.google.com.tr/citations?user=mKKkCFUAAAAJ&hl=tr`

Welcome Speech of Professor Vincent Bouhier
(President of the University of Évry Paris-Saclay)

Dear colleagues,

I am delighted to have been invited and to be present for the first day of the 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023).

It is an honor and a pleasure for the University of Évry Paris Saclay to host you.

I would also like to thank those who worked for this event, especially our colleague Abdelmejid BAYAD and all the administrative services that were involved.

The University of Évry Paris-Saclay is open to international research and it is its vocation to host scientific events such as MICOPAM 2023. Indeed, research is a crucial priority for the current governance, with efforts towards project funding, equipment, maintenance, and recruitment.

The University of Évry has existed **since 1991**. It has always managed to stay on course and regardless of the challenges it has faced, it has remained true to the values on which it was founded: **equal opportunities** through quality support for students, **open-mindedness** in the proposal of innovative educational tools, excellence in research by placing it at an international level, and finally, proximity.

The University of Évry is now home to over **160 programs** and **11500 students** from Bachelor's to Doctorate level, spread **across 14 campuses**. The university is constantly adapting to train future citizens and researchers with the necessary skills that society and the market demand. To achieve this, numerous bridges and partnerships are constantly being created with socio-economic actors, resulting in professional training programs in high-demand sectors.

In addition, the university houses **18 laboratories** and **9 technological platforms**. Today and in the future, the University of Évry is a place of experimentation and research with its own legal personality within the Paris-Saclay area. Beyond a label, this allows for the formation of a formidable force in education and research. This collective work is already internationally recognized, ranking **15th in the Shanghai 2023 ranking** (the top university in continental Europe).

Finally, the university is a meeting place. lt is a living space where we hope that all worlds, both public and private, intersect through the campus's openness to the city and the creation of new projects such as the GenoTher biocluster focused on gene therapy.

MICOPAM 2023 can rely on the laboratory of LaMME (Laboratory of Mathematics and Modeling of Évry), which is a UMR. Mathematics within the University of Paris-Saclay is a globally recognized field of research excellence, in which LaMME is directly involved. LaMME is supported by three research teams whose dynamism is undeniable:

- Analysis and Partial Differential Equations,
- Probability and Financial Mathematics,
- Statistics and Genomics.

Your presence today fully demonstrates our support for research and our desire to strengthen ties with researchers from other institutions. The richness of research lies in the meeting, exchange, and transmission.

I wish you fruitful work and an excellent stay.

Professor Vincent Bouhier
President of the University of Évry Paris-Saclay
Paris, France
Website: `https://www.univ-evry.fr`

Ceremony Talk of Professor Abdelmejid Bayad
(Head of the Organizing Committee)

Ladies and gentlemen,

Dear colleagues,

It is with immense pride and great honor that I stand before you today to inaugurate this **mathematics** conference. We are gathered here to celebrate and explore the wonders of this discipline "**mathematics**" that has shaped our world and continues to do so every day.

Mathematics is much more than just a science. It is a universal language that transcends borders and cultures. It is a discipline that allows us to understand and explain the most complex phenomena of our universe. **Mathematics** is omnipresent in our daily lives, whether it be in the technologies we use, the economic models we follow, or even in the arts and music.

Today, we also have the opportunity via our conference to pay tribute and **dedicate this conference to our colleague Professor YILMAZ SIMSEK**, whose name "**SIMSEK**" is synonymous with excellence, has devoted his life to the study and teaching of **mathematics**. His passion and expertise have inspired generations of students and researchers, and we are honored to dedicate this conference to him.

Professor **Yilmaz Simsek**, you are a role model for all of us. Your love for **mathematics** and perseverance have been a source of inspiration for many researchers. You have pushed the boundaries of knowledge and opened new paths in this field. Your dedication to teaching has also allowed many students to discover the beauty of mathematics and realize their full potential.

Today, we are gathered to share our knowledge, ideas, and discoveries. This conference is a unique opportunity to collaborate, learn from each other, and push the boundaries ot our understanding. **Mathematics** is a constantly evolving field, and it is through events like this one that we can continue to progress.

I would like to warmly thank the **President of our University forms support and his presence with us today and lot of thanks for his kind welcome speech**. Also thanks to all the **participants, speakers, and organizers** who have contributed to making this conference a reality. A special thanks to the **individuals, family and friends** who are accompanying you during this event. Your commitment and passion are essential to advancing our field.

In conclusion, I invite you to fully enjoy these days of sharing and exchange. May this conference be a source of inspiration and new perspectives for each of us. And let us not forget to pay tribute to **Professor Yilmaz Simsek**, whose legacy will continue to shine through our work and achievements.

I wish you alt an exciting conference.

Thank you.

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Dedicated to the respected mathematician
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on the occasion of his 60th anniversary

MICOPAM 2023

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The Proceedings Book of 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023)

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1 INVITED SPEAKERS

Model selection for protein copy numbers in populations of microorganism

Burcin Simsek [*1] *and Satish Iyengar* [2]

Recent biophysical studies have raised questions about the possible universality of protein copy number fluctuations. We are interested in comparing the fits of several models to those fluctuations. These models include the lognormal, generalized inverse Gaussian, and Frechet using closeness as measured by the Kullback-Leibler divergence. The lognormal results from a large number of multiplicative processes, or exponential growth; the generalized inverse Gaussian arises as a first passage time for diffusions; and the Frechet is an extreme value distribution. In this study, we show that the lognormal gives the best fit, and discuss implications for underlying biophysical processes.

2020 MSC: 53B20, 03C98, 60E05, 62E10, 60E10

KEYWORDS: Kullback-Leibler divergence criterion, Model fit, Generalized inverse Gaussian distribution, Frechet distribution, Lognormal distribution, Protein copy number fluctuations data

Scope and detail of the presentation

Recent biophysical studies have raised questions about the possible universality of protein copy number fluctuations. Salman (2015) have shown that the shape of the protein copy number distribution measured for different proteins in two different microorganisms exhibits universal features. Our main goal here is to compare the fits of lognormal, generalized inverse Gaussian and Frechet distributions to protein copy number fluctuations data using formal statistical model selection procedures. Closeness of two density functions are measured by the Kullback-Leibler divergence criterion, and lognormal distribution yield the best fit for the underlying biophysical processes.

Acknowledgments

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This presentation is dedicated to the 60th birthday of Prof. Dr. Yilmaz Simsek.

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S. Ramanujan - the mathematician and his legacy

Chandrashekar Adiga

Self-taught mathematical prodigy Srinivasa Ramanujan had a brilliant but short life. He was born in the home of his maternal grand parents on 22, December 1887 in Erode. In 1920, at the age of 32, he died from a combination of illness and malnutrition. Influenced by Carr's book "A Synopsis of Elementary results in Pure Mathematics", Ramanujan started recording statements of his theorems without proofs in his notebooks. There are three famous notebooks written by Ramanujan. These books have inspired mathematicians ever since and have formed the basis for numerous papers for many mathematicians including Hardy, Watson, Rogers, Bruce Berndt, G. E. Andrews, Richard Askey, R. P. Agarwal, S. Bhargava and many others. In this survey article, we will briefly sketch the life of Ramanujan and some of his important contributions. During 1914-1917, Ramanujan colloborated with G. H. Hardy and wrote several important papers. The most notable of these colloborations involved the partition function $p(n)$. Ramanujan discovered some elegant congruence properties of partition function. Hardy and Ramanujan developed a new method, now called the circle method, to derive an asymptotic formula for $p(n)$. This method is now one of the important tools of analytic number theory. In fact this method is mainly responsible for major advances of problems such as Goldbach's conjecture, Waring's conjecture and other additive number thoery problems. In Chapter 16 of his second notebook, Ramanujan discovered several new q-series identities and developed the theory of theta-functions. One of the beautiful q-series identities discovered by Ramanujan is now well-known as "Ramanujan's summation of the ${}_1\psi_1$". Rogers-Ramanujan identities, undoubtedly the most famous of Ramanujan's discoveries in the area of q-series are also found in Chapter 16. Ramanujan found several infinite series representations for pi. In 1916, Ramanujan studied another important function called Ramanujan tau function $\tau(n)$ which is multiplicative. Lehmer (1947) conjectured that $\tau(n) \neq 0$ for all n. Ramanujan made the following amazing conjecture: **Conjecture:** For all primes p,

$$|\tau(p)| \leq 2p^{11/2}.$$

This was proved by Deligne, for which he got the Fields Medal in 1978. While he was in deathbed, Ramanujan discovered "mock theta functions". In 1987, the famous physicist Freeman Dyson predicted: "The mock theta functions give us tantalising hints of grand synthesis still to be discovered. It should be possible to build them into coherent group-theoretical structure, anologus to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains the challenge for the future".

2020 MSC: 11P81, 11P83, 11P84, 11P65, 33D90

KEYWORDS: Partition function, q-series, Mock theta function, Ramanujan tau function, Divisor function

Acknowledgments

This article is dedicated to Prof. Yilmaz Simsek on the occasion of his 60th birthday.

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Convective heat transfer in an enclosure in the presence of a magnetic field

Ceeke Kalappagowda Subbaraya

Natural convection in a rectangular enclosure with differentially heated side walls and insulated horizontal surface has been the subject of a great number of experimental and theoretical investigations. This is due to its applications in solar technology, safety aspects of gas cooled reactors and crystal growth in liquids. The work reported in these studies pertain to convection in an ordinary fluid. The process of manufacturing materials in industrial problems (for example, crystal growth using the horizontal Bridgman technique) involve an electrically conducting fluid subjected to a magnetic field. In that case the fluid experiences a Lorentz force and its effect is to reduce the velocities. This in turn affects the rate of heat and mass transfer. It is, therefore, important to study the detailed characteristics of transport phenomena in such a process so that good quality products can be developed with improved design in the manufacturing processes. The aim of the present work is to examine in detail the effect of a magnetic field on natural convection and heat transfer in a two-dimensional enclosure filled with an electrically conducting fluid. The applied magnetic field is assumed to be parallel to the gravity. The two vertical boundary surfaces of the enclosure are maintained isothermal, with one of the surfaces being heated and other being cooled. Adiabatic conditions are imposed at the two horizontal bounding surfaces. For many fluids used in the laboratory the conductivity is usually small and hence the magnetic Reynolds number is very small. Therefore, we assume that the induced magnetic field produced by the motion of the electrically conducting fluid is negligible compared to the applied magnetic field B0. Then the electromagnetic retarding force and the buoyancy force terms appear in the horizontal and vertical momentum equations respectively with a result that a boundary layer type of analysis is not applicable. Thus, the equations are nonlinear and coupled in nature and hence not amenable to analytical treatment. We, therefore, solve the problem numerically using an implicit finite difference scheme which is computationally stable. The effect of various controlling parameters on fluid flow and heat transfer are examined.

2020 MSC: 76R10, 76W05, 80A20

Keywords: Natural convection, Rectangular enclosure, Differentially heated side walls, Electrically conducting fluid, Magnetic reynolds number

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Iterated sociable number and stable numbers

Daeyeoul Kim

This paper is a paper commemorating the 60th birthday of Professor Yilmaz Simsek. We would like to explain the most basic iterated number and introduce related results.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Generalizations of the Hardy type inequality via new Green functions and interpolating polynomials

Dora Pokaz

We start with the Hardy type inequality in the general setting given in [2] that is considering measure spaces $(\Sigma_i, \Omega_i, \mu_i)$, $i = 1, 2$, with positive σ-finite measures, a measurable and non-negative kernel $k : \Omega_1 \times \Omega_2 \to [0, \infty)$ such that

$$0 < K(x) = \int_{\Omega_2} k(x,t) d\mu_2(t), x \in \Omega_1.$$

For a measurable function $f \colon \Omega_2 \to \mathbf{R}$, with A_k we denote the integral operator

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t).$$

Let the weight function $u : \Omega_1 \to [0, \infty)$ and kernel $k : \Omega_1 \times \Omega_2 \to [0, \infty)$ are such that $\frac{k(x,y)}{K(x)} u(x)$ is locally integrable on Ω_1 for each $y \in \Omega_2$ and let v be given by

$$v(y) = \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty.$$

Considering the above settings, in [3] we have proved that for a convex function ϕ on an interval $I \subseteq \mathbf{R}$, the following inequality

$$\int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \leq \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y)$$

holds for all measurable functions $f : \Omega_2 \to I$.

Based on the article [4] with $\tilde{G}_\gamma$, $\gamma = 1, 2, 3, 4$, we will denote the following Green functions defined on $[\alpha, \beta] \times [\alpha, \beta]$ with

$$\tilde{G}_1(t,s) = \begin{cases} \alpha - s, & \alpha \leq s \leq t; \\ \alpha - t, & t \leq s \leq \beta. \end{cases}$$

$$\tilde{G}_2(t,s) = \begin{cases} t - \beta\ , & \alpha \leq s \leq t; \\ s - \beta, & t \leq s \leq \beta. \end{cases}$$

$$\tilde{G}_3(t,s) = \begin{cases} t - \alpha\ , & \alpha \leq s \leq t; \\ s - \alpha, & t \leq s \leq \beta. \end{cases}$$

$$\tilde{G}_4(t,s) = \begin{cases} \beta - s\ , & \alpha \leq s \leq t; \\ \beta - t, & t \leq s \leq \beta. \end{cases}$$

In this talk we will present new results regarding Hardy-type inequality for given general settings. A connection between the difference operator, obtained from Hardy-type inequality, and the expression that includes interpolating polynomial such as Lidstone or Abel-Gontscharoff as well as four new Green functions will be established. Some results concerning the Abel-Gontscharoff interpolating polynomial can be found in [5]. Further, we will discuss about the parity of the

index n and the n- convexity of the function. We get some consequential results applying Hölder inequality for conjugate exponents p and g. As it is common, we will conclude with Čebyšev functional and Ostrowski-type (see [1]) bound derived for the generalized Hardy's inequality.

2020 MSC: 26D10, 26D15, 26B20

KEYWORDS: Convex function, Upper bounds, Hardy type inequality, Abel–Gontscharoff interpolating polynomial, Lidstone interpolating polynomial, Green function, new Green functions, Čebyšev functional

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Extensions of valuations with greater rank

Figen Oke

If v be a valuation on a field K, w is a residual algebraic free extension of v to $K(x)$ then $rankw > rankv$. In this paper certain residual algebraic free extensiions of a valuation v of K to $K(x)$ is described when $rankv = 2$.

2020 MSC: 12F051, 12J10, 12J20

KEYWORDS: Valuations, Residual algebraic free extensions, Extensions of valuations

Introduction

Defining all extensions of a valuation on a field K to the rational function field $K(x_1, ..., x_n)$ is an old and important problem. The first step of this problem is defining the extensions of a valuation on K to $K(x)$. The residual transcendental extensions of a valuation on a field K to $K(x)$ was studied in [1, 2] and all extensions were introduced in [3]. In this paper it is assumed that v is a valuation on a field K with $rankv = 2$ and certain residual algebraic extensions of v to $K(x)$ are studied by considering valuation rings and value groups. This subject was introduced by the author in [6] and a kind of residual algebraic free extension of a valuation v of a field K to $K(x)$ was studied, where $rankv = 2$.

Preliminaries

Throughout this paper, v is a valuation of K with value group G_v, residue field k_v, valuation ring O_v and $\bar{v}$ is the fixed extension of v to the algebraic closure $\bar{K}$ of K. For any λ in the valuation ring of v, λ_v^* will denote its $v-$ residue, i.e. the image of λ under the canonical homomorphism from the valuation ring of v onto residue field k_v. If λ isn't an element of valuation ring then for an element $b \in K$ such that $v(\lambda) = v(b)$, λ_v^* will denote the $v-$ residue of λ/b.

If k_w/k_v is a residual transcendental extension, w is called a residual transcendental extension of v to $K(x)$ and it is defined by minimal pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$, $k_w = k_{v_a}(r^*) = k_{v_a}(Y)$, writing $r^* = Y$, where $r = \frac{f^e}{h} \in K(x)$, $w(r) = 0$, r^* is transcendental over residue field k_{v_a} , e is the smallest positive integer such that $e\gamma \in G_{v_a}$ and $h(x) \in K[x]$ such that $\deg h < \deg f$ (*cf.* [1], [2], [3]).

If G is the lifting polynomial in $K[x]$ of $g \neq Y$, w is defined for each $F \in K[x]$, $F = F_0 + F_1G + ... + F_nG^n$, $\deg F_i < \deg G$, $i = 0, ..., n$ as $w(F) = \inf_i((w_1(F_i), 0) + i(w(G), 1))$ is a residual algebraic free extension of v to $K(x)$ with value group $G_{\overline{v}}\mathrm{x}Q$ which is ordered lexicographically and with residue field $\mathrm{k}_{\mathrm{v_b}}$, where b is the suitable root of G (*cf.* [8]).

Main results

Certain residual algebraic free extensions of a valuation v on a field K such that $rankv = 2$ are described:

Theorem 1. *Let (K, v) be a valued field and $w = w_1 \circ w_2 \circ w_3$ be a residual algebraic free extension of v to the field $K(x)$ such that $rankw = 3$. If $O_{w_1} \cap K = K$ then w is a first kind residual algebraic free extension of v to $K(x)$. w_1 is defined by an irreducible polynomial $f \in K[x]$ and $w_2 \circ w_3$ is an extension of v to $K' = k_{v_1}$ which is an algebraic extension of K. Then for each polynomial $F \in K[x]$, $F = F_0 + F_1 f + ... + F_n f^n$, $\deg F_i < \deg f$, $i = 0, ..., n$, w is defined as;*
$w(F) = (w_1 \circ w_2 \circ w_3)(F)$
$= \inf_i((i, 0, 0) + (0, w_2(F_i(a)), 0) + (0, 0, w_3(p_{w_2}(F_i(a)/c_{i_k} a^k))),$
where $w_2(F_i(a)) = w_2(c_{i_0} + c_{i_1} a + ... + c_{i_n} a^n) = \inf_i w_2(c_{i_j} a^j) = w_2(c_{i_k} a^k)$, where a is a root of f, w_2 is an extension of v_1 to $k_{w_1} = K(a)$ and w_3 is an extension of v_2 to k_{w_2}.

The other kind r.a.f extension of v to $K(x)$ given in the following theorem:

Theorem 2. *Let $v = v_1 \circ v_2$ be a valuation with $rankv = 2$ and $w = w_1 \circ w_2 \circ w_3$ be a r.a.f extension of v to $K(x)$ with $rankw = 3$. Denote $u = w_1 \circ w_2$. If $O_u \cap K = O_v$ then u is r.t. extension of v to $K(x)$ and w is a second kind r.a.f extension of v to $K(x)$ and it is defined by a minimal pair (c, λ) with respect to (K, v) using the way introduced in [3, 8]. Here it is important that w_1 is r.t. extension of v_1 and w_2 is r.t. extension of v_2.*

Conclusion

In this study certain residual algebraic free extensions of a valuation v of a field K to $K(x)$ is given, where $rankv = 2$.

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Complex orthogonality on the semicircle and numerical integration and differentiation

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Complex orthogonality on the semicircle with respect to a complex moment functional (or to the non-Hermitian inner product) was introduced and studied by Gautschi and Milovanović (1986), and later generalized together with Landau (1987). Milovanović (1989) gave a generalization with respect to the Gegenbauer weight function. Beside this background, starting from our recent results on the Laurent orthogonal functions on the semicircle (see [*Bull. Cl. Sci. Math. Nat. Sci. Math.* **44**, 1–28, 2019] and [*Electron. Trans. Numer. Anal.* **59**, 99–115, 2023]) we present new results on numerical integration and numerical differentiation.

2020 MSC: 30C10, 30C15, 33C47, 42C05, 65D25, 65D30

KEYWORDS: Complex orthogonal systems, Orthogonal Laurent polynomials, Recurrence relations, Quadrature formula, Numerical differentiation, Zeros

Introduction

Let $\mathcal{P}_n$ be the space of all algebraic polynomials of degree at most n, $\mathcal{P}$ be the space of all polynomials, and $\{\pi_k\}$ be a system of monic orthogonal polynomials with respect to a given inner product $(\,\cdot\,,\,\cdot)$.

Two standard types of orthogonal polynomials are: (a) *polynomials orthogonal on the real line*, when the inner product is given by

$$(p,q)=\int_a^b p(x)q(x)w(x)\,\mathrm{d}x \quad (p,q\in\mathcal{P}),$$

where $(a,b)\subset\mathbb{R}$ and w is a given weight function; (b) *polynomials orthogonal on the unit circle*, when

$$(p,q)=\int_{-\pi}^{\pi} p(\mathrm{e}^{\mathrm{i}\theta})\overline{q(\mathrm{e}^{\mathrm{i}\theta})}w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \quad (p,q\in\mathcal{P}).$$

However, Gautschi and Milovanović [2] introduced (and later generalized together with Landau [3]) orthogonal polynomials on the semicircle with respect to the quasi-inner product given by

$$\langle p,q\rangle=\int_0^{\pi} p(\mathrm{e}^{\mathrm{i}\theta})q(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \quad (p,q\in\mathcal{P}),$$

where the second factor is not conjugated, so that this product is not Hermitian. Some applications were given in [4] and [1].

Precisely, this concept of orthogonality can be treated in general with respect to a *complex moment functional*

$$\mathcal{L}[z^k]=\mu_k=\langle 1,z^k\rangle=\int_0^{\pi}\mathrm{e}^{\mathrm{i}k\theta}w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta,\quad k=0,1,2,\ldots\,. \tag{1}$$

The corresponding (monic) orthogonal polynomials π_k exist uniquely under the mild restriction $\operatorname{Re}\mu_0 = \operatorname{Re}\int_0^\pi w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \neq 0$ (see [3]). The case with the Gegenbauer weight $w(z) = w^\lambda(z) = (1-z^2)^{\lambda-1/2}$, $\lambda > -1/2$, was considered in [8]. In this case, the monic orthogonal polynomials $\{\pi_k^\lambda\}$ on the semicircle can be expressed in terms of the monic Gegenbauer polynomials $\widehat{C}_k(z)$ as $\pi_k^\lambda(z) = \widehat{C}_k^\lambda(z) - \mathrm{i}\theta_{k-1}\widehat{C}_{k-1}^\lambda(z)$, where the sequence $\{\theta_k\}$ is given recursively by

$$\theta_0 = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\lambda+1)}\,, \quad \theta_k = \frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}\cdot\frac{1}{\theta_{k-1}}\,, \quad k = 1,2,\ldots\,,$$

wherefrom we obtain the following explicit form in terms of the gamma function,

$$\theta_k = \frac{1}{\lambda+k}\cdot\frac{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\lambda+\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\lambda+\frac{k}{2}\right)}\,, \quad k \geq 0\,.$$

Theorem 1. *If $\lambda > -1/2$, then all zeros of the orthogonal polynomials $\pi_k^\lambda(z)$, $k \geq 2$, are simple, distributed symmetrically with respect to the imaginary axis, and contained in the open upper unit half disk $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$.*

Preliminaries

Recently we considered the same concept of orthogonality (1), which includes positive and negative exponents, i.e., $-n+1 \leq k \leq n$ (see [9, 10]).

Let $\Lambda_{p,q}$ be a linear space, over the complex numbers $\mathbb{C}$, of polynomials generated by the basis $\mathcal{B}_{p,q} = \{z^p, z^{p+1}, \ldots, z^q\}$, where $p, q \in \mathbb{Z}$ and $p \leq q$. For $p = 0$ and $q = n \in \mathbb{N}_0$ this space reduces to the space of polynomials $\Lambda_{0,n} = \mathcal{P}_n$.

Applying the Gram-Schmidt orthogonalization process to the sequence of monomials $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$ we obtain a system of orthogonal elements in $\Lambda_{-n+1,n}$, in notation $\{R_\nu\}$. Such an orthogonal system is known as the *Laurent system of orthogonal polynomials* (see [9, 10]).

Theorem 2 (*cf.* [10]). *Let the complex moment functional be given by* (1)*, where $k = 0, \pm1, \pm2, \ldots$, and the corresponding Laurent polynomials $R_m(z)$ be orthogonal in $\Lambda_\pm$ satisfying*

$$\langle R_n, R_m\rangle = \int_0^\pi R_n(\mathrm{e}^{\mathrm{i}\theta})R_m(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \|R_n\|^2\delta_{n,m} \quad (n, m \in \mathbb{N}_0),$$

where $\delta_{n,m}$ is Kronecker's delta, then the following two three-term recurrence relations

$$R_{2k+1}(z) = (z - a_{2k})R_{2k}(z) + b_{2k}R_{2k-1}(z),$$

$$R_{2k+2}(z) = \left(1 - \frac{a_{2k+1}}{z}\right)R_{2k+1}(z) + b_{2k+1}R_{2k}(z),$$

hold, where $R_0(z) = 1$ and $R_{-1}(z) = 0$, and $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers depending only on the weight function $w(z)$.

Since $R_m(z) \in \Lambda_{-[m/2],[(m+1)/2]}$ $(m \in \mathbb{N}_0)$ it can be represented as

$$R_m(z) = \sum_{\nu=0}^m c_\nu^{(m)} z^{\nu-[m/2]} = \frac{1}{z^{[m/2]}}\sum_{\nu=0}^m c_\nu^{(m)} z^\nu, \quad m \in \mathbb{N}_0,$$

for some constants $c_\nu^{(m)}$, $\nu = 0, 1, \ldots, m$. Putting $Q_m(z) = z^{[m/2]}R_m(z)$, for these numerators of the Laurent orthogonal polynomials, we have the following recurrence relation

$$Q_{k+1}(z) = (z - a_k)Q_k(z) + b_k z\, Q_{k-1}(z), \quad k = 0, 1, \ldots, \tag{2}$$

with $Q_0(z) = 1$ and $Q_{-1}(z) = 0$, where the recurrence coefficients are given as in Theorem 2. These polynomials $Q_m(z)$ can be characterized by the following orthogonality relations

$$\int_0^\pi \mathrm{e}^{-\mathrm{i}k\theta} Q_m\big(\mathrm{e}^{\mathrm{i}\theta}\big) w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta = 0, \quad k = 0, 1, \ldots, m-1.$$

Three interesting symmetric cases, with the Gegenbauer weight functions: (1) Legendre weight $w(z) = 1$ ($\lambda = 1/2$); (2) Chebyshev weight of the first kind $w(z) = (1-z^2)^{-1/2}$ ($\lambda = 0$); (3) Chebyshev weight of the second kind $w(z) = (1-z^2)^{1/2}$ ($\lambda = 1$), have been recently considered in [9, 10]. For $\lambda > 0$ all zeros of the corresponding polynomials $Q_k(z) \equiv Q_k^\lambda(z)$, $k \in \mathbb{N}$, belong to the open upper unit half disk $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$, but in the Chebyshev case of the first kind ($\lambda = 0$), these zeros are symmetrically distributed on the upper semicircle and can be expressed explicitly in the form

$$\zeta_k = \cos\varphi_k \sqrt{1 + \sin^2\varphi_k} + \mathrm{i}\sin^2\varphi_k = \mathrm{e}^{\mathrm{i}\theta_k}, \quad k = 1, 2, \ldots, n,$$

where $\varphi_k = (2k-1)\pi/(2n)$, $\sin\theta_k = \sin^2\varphi_k$, $k = 1, 2, \ldots, n$ (see [10, Thm. 4.3]).

Main results

In this section, we present the main results on quadrature formulae with respect to the even weight function on the semicircle, which are exact on the space of Laurent polynomials $\Lambda_{-n+1,n}$.

Theorem 3. *The quadrature formula*

$$\int_0^\pi F\big(\mathrm{e}^{\mathrm{i}\theta}\big) w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta = \sum_{k=1}^n A_k F(\zeta_k) + E_n(F) \tag{3}$$

is exact for each $F \in \Lambda_{-n+1,n}$ *if and only if its nodes* $\zeta_k = \zeta_k^{(n)}$ *are zeros of the polynomial* $Q_n(z)$*, and the weight coefficients* A_ν *are given by*

$$A_k = A_k^{(n)} = \frac{1}{Q_n'(\zeta_k)} \int_0^\pi \frac{Q_n\big(\mathrm{e}^{\mathrm{i}\theta}\big) w\big(\mathrm{e}^{\mathrm{i}\theta}\big)}{\mathrm{e}^{\mathrm{i}\theta} - \zeta_k}\,\mathrm{d}\theta, \quad k = 1, \ldots, n. \tag{4}$$

Quadrature formulas (3) can be successfully applied to the calculation of quasi-singular integrals on the interval $(-1, 1)$, e.g., for integrals of the form

$$\int_{-1}^1 \frac{f(x)w(x)}{\big[(x-c)^2 + \varepsilon^2\big]^{r/2}}\,\mathrm{d}x, \quad -1 < c < 1,\ |\varepsilon] \ll 1,\ r \in \mathbb{N},$$

which appear in some fields of engineering. Another important application is in numerical differentiation.

There are several finite difference formulas for numerical differentiation of a function $z \mapsto f(z)$ and their application in solving differential equations. For details, see the three-volume book Numerical Analysis [5, 6, 7].

Let f be an analytic function in certain domain containing the point a and its circular neighborhood, with the center at a and radius r. Using the central difference operator δ_h, defined by

$$\delta_h f(a) = \frac{1}{h}\left(f\left(a + \frac{h}{2}\right) - f\left(a - \frac{h}{2}\right)\right),$$

with the error $O(h^2)$, and putting $he^{i\theta}$ instead of h, where h is such that $\left|a+\frac{h}{2}e^{i\theta}\right| < r$, we get

$$\int_0^\pi \delta_{he^{i\theta}} f(a) w(e^{i\theta})\, d\theta = \pi f'(a).$$

Now, applying the quadrature formula (3), we get the following differentiation formula for the first derivative

$$f'(a) \approx D_{n,h} f(a) = \frac{1}{\pi} \sum_{\nu=1}^{n} A_\nu \delta_{h\zeta_\nu} f(a),$$

i.e.,

$$D_{n,h} f(a) = \frac{1}{\pi h} \sum_{\nu=1}^{n} \frac{A_\nu}{\zeta_\nu} \left[f\left(a + \frac{h}{2}\zeta_\nu\right) - f\left(a - \frac{h}{2}\zeta_\nu\right)\right], \tag{5}$$

where the coefficients A_ν are given in (4). Usually we take $n = 2$, when for real-valued functions and $a \in \mathbb{R}$, the formula (5) reduces to

$$D_{2,h} f(a) = \frac{1}{\pi h} \mathrm{Re} \left\{ \frac{A_1}{\zeta_1} \left[f\left(a + \frac{h}{2}\zeta_1\right) - f\left(a - \frac{h}{2}\zeta_1\right)\right]\right\}.$$

Then, the error of the last formula is $O(h^4)$.

Similar consideration can be done for higher derivatives.

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Integer partitions, graphs and monomial ideals

Hussein Mourtada

We will show two new partition identities which are dual to the Rogers-Ramanujan identities. These identities are inspired by a correspondence between three kinds of objects: a new type of partitions (neighborly partitions), monomial ideals and some infinite graphs. This talk is based on a joint work with Z. Mohsen [1].

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KEYWORDS: Integer partitions, Rogers-Ramanujan identities, Graphs, Monomial ideals

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On Vietoris convolution triangles

Isabel Cação

We consider the so-called Vietoris number sequence, a sequence of rational numbers that appeared for the first time in a celebrated theorem by L. Vietoris (1958) about the positivity of certain trigonometric sums. In this work we construct convolution triangles associated with that sequence and compute determinants of matrices related with the corresponding convolution triangles.

2020 MSC: 11B83, 11B65, 30G35, 05A10, 33C10

Keywords: Vietoris number sequence, Central binomial coefficients, Convolution triangles

Introduction

The so-called Vietoris' number sequence $(a_k)_{k\geq 0}$ is a sequence of rational numbers that appeared for the first time in a celebrated paper of L. Vietoris [10] in the context of positive trigonometric sums:

Theorem 1. *Let* $a_{2k} = a_{2k+1} = \frac{1}{2^{2k}}\binom{2k}{k}$, $k = 0, 1, 2, \ldots$. *Then,*

$$\sum_{k=1}^{n} a_k \sin(k\theta) > 0 \;\; \textit{for} \;\; 0 < \theta < \pi,$$

and

$$\sum_{k=0}^{n} a_k \cos(k\theta) > 0 \;\; \textit{for} \;\; 0 \leq \theta < \pi.$$

Since then, it has appeared in other contexts such as harmonic analysis [1], in the theory of stable holomorphic functions [9], and more recently in hypercomplex function theory [3]. In this last context, the sequence $(a_k)_{k\geq 1}$ that appears in the sine sum of the Theorem 1 plays a crucial role in the generalization of Appell polynomials to arbitrary dimensions in the framework of Clifford Algebras [7]. A suitable analogue for the holomorphic powers in higher dimensions within Clifford analysis is a class of hyperholomorphic Appell polynomials whose coefficients are precisely the terms of the Vietoris number sequence $(c_n)_{n\geq 0}$, defined in Theorem 1 and such that $c_n := a_{n+1}$, i.e.

$$c_0 = 1, \;\; c_{2n} = c_{2n-1} = \frac{1}{2^{2n}}\binom{2n}{n}, \; n = 1, 2, \ldots. \tag{1}$$

Explicitely, the first terms of this sequence are

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \ldots .$$

From (1) we recognize immediately the connection of the subsequence of the even terms (or odd terms) and the central binomial coefficients sequence $\left(v_n := \binom{2n}{n}\right)_{n\geq 0}$,

$$c_{2n} = \frac{1}{2^{2n}} v_n,\ n = 0, 1, 2, \ldots. \tag{2}$$

Using the floor function $\lfloor \cdot \rfloor$, a unique representation of the general term of the Vietoris number sequence $(c_n)_n$ with consecutively running index n is obtained:

$$c_n = \frac{1}{2^n}\binom{n}{\lfloor \frac{n}{2} \rfloor},\ n = 0, 1, 2, \ldots. \tag{3}$$

The representation (3) shows an obvious link to the sequence of *complete* central binomial coefficients $\left(u_n := \binom{n}{\lfloor \frac{n}{2} \rfloor})_n\right)_{n\geq 0}$,

$$c_n = \frac{1}{2^n} u_n,\ n = 0, 1, 2, \ldots. \tag{4}$$

An elementary procedure based on the Taylor series of the binomial functions

$$f_m(t) = \tfrac{1}{(1-t)^m},\ m = 1, 2, \ldots \tag{5}$$

derives the following generating function of the Vietoris sequence $(c_n)_n$:

Theorem 2 (*cf.* [3])**.** *It holds*

$$F(t) := \frac{2}{1 - t + \sqrt{1-t^2}} = \sum_{n=0}^{\infty} c_n t^n, \quad t \in]-1, 1[\setminus\{0\}, \tag{6}$$

where c_n are given by (3)*.*

As a consequence, we can establish the following trigonometric identities:

Corollary 3 (*cf.* [3])**.** *For $0 < \alpha < \pi$ and c_n $(n = 0, 1, \ldots)$ given by* (3)*, we have*

$$\sum_{n=0}^{\infty} c_n \cos^n \alpha = \frac{2}{1 - \cos\alpha + \sin\alpha} \quad \textit{and} \quad \sum_{n=0}^{\infty} c_n \sin^n \alpha = \frac{2}{1 - \sin\alpha + \cos\alpha},$$

For a detailed study about other properties of the Vietoris sequence and generalizations we refer [4, 5, 6].

Vietoris convolution triangles and related determinants

We start this section by recalling the definition of a sequence's convolution with itself.

Definition 4 (*cf.* [8])**.** *Given a sequence $(a_n)_n$, its k-th convolution is the sequence $(a_n^{(k)})_n$, defined recursively as*

$$a_n^{(k)} = \sum_{s=0}^{n} a_s a_{n-s}^{(k-1)}, \quad k = 1, 2, \ldots$$
$$a_n^{(0)} = a_n.$$

Table 1: The Vietoris convolution triangle $\mathcal{T}_V$

1	1	1	1	1	1	1	$\dots$	1						
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	$\dots$	$\frac{1}{2}$	1					
$\frac{1}{2}$	$\frac{5}{4}$	$\frac{9}{4}$	$\frac{7}{2}$	5	$\frac{27}{4}$	$\frac{35}{4}$	$\dots$	$\frac{1}{2}$	1	1				
$\frac{3}{8}$	$\frac{5}{4}$	$\frac{11}{4}$	5	$\frac{65}{8}$	$\frac{49}{4}$	$\frac{35}{2}$	$\dots$	$\frac{3}{8}$	$\frac{5}{4}$	$\frac{3}{2}$	1			
$\frac{3}{8}$	$\frac{11}{8}$	$\frac{27}{8}$	$\frac{109}{16}$	$\frac{195}{16}$	$\frac{321}{16}$	$\frac{497}{16}$	$\dots$	$\frac{3}{8}$	$\frac{5}{4}$	$\frac{9}{4}$	2	1		
$\frac{5}{16}$	$\frac{11}{8}$	$\frac{123}{32}$	$\frac{69}{8}$	$\frac{541}{32}$	$\frac{483}{16}$	$\frac{805}{16}$	$\dots$	$\frac{5}{16}$	$\frac{11}{8}$	$\frac{11}{4}$	$\frac{7}{2}$	$\frac{5}{2}$	1	
$\frac{5}{16}$	$\frac{93}{64}$	$\frac{281}{64}$	$\frac{341}{32}$	$\frac{45}{2}$	$\frac{689}{16}$	$\frac{1225}{16}$	$\dots$	$\frac{5}{16}$	$\frac{11}{8}$	$\frac{27}{8}$	5	5	3	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The convolution of sequences corresponds to the multiplication of their generating functions, i.e., being $F(t)$ a generating function of the sequence $(c_n)_n$, then a generating function $F_k(t)$ of its k-th convolution $(c_n^{(k)})_n$ is $(F(t))^{k+1}$, $k = 0, 1, 2, \dots$ (see e.g. [2]).

Example 5. *From the generating function* (6)*, we obtain*

$$\begin{aligned}
F_0(t) =& \frac{2}{1-t+\sqrt{1-t^2}} \\
=& \mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}t + \frac{\mathbf{1}}{\mathbf{2}}t^2 + \frac{\mathbf{3}}{\mathbf{8}}t^3 + \frac{\mathbf{3}}{\mathbf{8}}t^4 + \frac{\mathbf{5}}{\mathbf{16}}t^5 + \frac{\mathbf{5}}{\mathbf{16}}t^6 + \frac{\mathbf{35}}{\mathbf{128}}t^7 + \frac{\mathbf{35}}{\mathbf{128}}t^8 + \dots \\
F_1(t) =& \frac{4}{\left(1-t+\sqrt{1-t^2}\right)^2} \\
=& \mathbf{1} + \mathbf{1}t + \frac{\mathbf{5}}{\mathbf{4}}t^2 + \frac{\mathbf{5}}{\mathbf{4}}t^3 + \frac{\mathbf{11}}{\mathbf{8}}t^4 + \frac{\mathbf{11}}{\mathbf{8}}t^5 + \frac{\mathbf{93}}{\mathbf{64}}t^6 + \frac{\mathbf{93}}{\mathbf{64}}t^7 + \frac{\mathbf{193}}{\mathbf{128}}t^8 + \dots \\
F_2(t) =& \frac{8}{\left(1-t+\sqrt{1-t^2}\right)^3} \\
=& \mathbf{1} + \frac{\mathbf{3}}{\mathbf{2}}t + \frac{\mathbf{9}}{\mathbf{4}}t^2 + \frac{\mathbf{11}}{\mathbf{4}}t^3 + \frac{\mathbf{27}}{\mathbf{8}}t^4 + \frac{\mathbf{123}}{\mathbf{32}}t^5 + \frac{\mathbf{281}}{\mathbf{64}}t^6 + \frac{\mathbf{309}}{\mathbf{64}}t^7 + \frac{\mathbf{681}}{\mathbf{128}}t^8 + \dots
\end{aligned}$$

Before proceeding to the construction of the convolution triangles associated with the Vietoris number sequence $(c_n)_n$ and its even order-term subsequence $(c_{2n})_n$, we recall the definiton of a convolution triangle associated with a number sequence.

Definition 6 (*cf.* [2])**.** *Given a sequence of numbers $(a_n)_n$, a convolution triangle is an array whose k-th column is the $(k-1)$-th convolution $(a_n^{(k-1)})_n$ of the sequence $(a_n)_n$ by itself.*

Tables 1-2 show the convolution triangles $\mathcal{T}_V$ and $\mathcal{T}_{EV}$ corresponding to the sequences $(c_n)_n$ and $(c_{2n})_n$, respectively. For the sake of better visibility, we write on the right side of each table the corresponding left justified triangle.

We consider now matrices associated with each convolution triangle and compute their determinants.

Table 2: The even order-terms Vietoris convolution triangle $\mathcal{T}_{EV}$

1	$\mathbf{1}$	1	$\mathbf{1}$	1	$\mathbf{1}$	1	$\dots$		1						
$\frac{1}{2}$	$\mathbf{1}$	$\frac{3}{2}$	$\mathbf{2}$	$\frac{5}{2}$	$\mathbf{3}$	$\frac{7}{2}$	$\dots$		$\frac{1}{2}$	1					
$\frac{3}{8}$	$\mathbf{1}$	$\frac{15}{8}$	$\mathbf{3}$	$\frac{35}{8}$	$\mathbf{6}$	$\frac{63}{8}$	$\dots$		$\frac{3}{8}$	$\mathbf{1}$	1				
$\frac{5}{16}$	$\mathbf{1}$	$\frac{35}{16}$	$\mathbf{4}$	$\frac{105}{8}$	$\mathbf{10}$	$\frac{231}{16}$	$\dots$		$\frac{5}{16}$	$\mathbf{1}$	$\frac{3}{2}$	$\mathbf{1}$			
$\frac{35}{128}$	$\mathbf{1}$	$\frac{315}{128}$	$\mathbf{5}$	$\frac{1155}{128}$	$\mathbf{15}$	$\frac{3003}{128}$	$\dots$		$\frac{35}{128}$	$\mathbf{1}$	$\frac{15}{8}$	$\mathbf{2}$	$\mathbf{1}$		
$\frac{63}{256}$	$\mathbf{1}$	$\frac{693}{256}$	$\mathbf{6}$	$\frac{3003}{256}$	$\mathbf{21}$	$\frac{9009}{256}$	$\dots$		$\frac{63}{256}$	$\mathbf{1}$	$\frac{35}{16}$	$\mathbf{3}$	$\frac{5}{2}$	$\mathbf{1}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Theorem 7. *Let $\mathcal{M}_V^{r\times r}$ be the $r \times r$ matrix formed by the first r rows and r columns of the triangle $\mathcal{T}_V$ written in its rectangular form. Then,*

$$\det \mathcal{M}_V^{r\times r} = 2^{-\frac{r(r-1)}{2}}.$$

Proof. From (4), the complete central binomial coefficients sequence $(u_n)_n$ and the Vietoris number sequence $(c_n)_n$ are linked by

$$u_n = 2^n\, c_n,\ n = 0, 1, 2, \dots.$$

This implies that the k-th convolution of the sequence $(u_n)_n$ by itself is simply expressed as

$$u_n^{(k)} = 2^n\, c_n^{(k)},\ \ k = 0, 1, 2, \dots,\ n = 0, 1, 2, \dots.$$

Let $\mathcal{M}_{CC}^{r\times r}$ be the matrix formed by the first r rows and r columns of the convolution triangle of the sequence $(u_n^{(k-1)})_n$. Then,

$$\mathcal{M}_{CC}^{r\times r} = \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 2^2 & & \\ & & & \ddots & \\ & & & & 2^{r-1} \end{pmatrix} \mathcal{M}_V^{r\times r}.$$

Consequently,

$$\mathcal{M}_{CC}^{r\times r} = \prod_{i=1}^{r} 2^{i-1} \det \mathcal{M}_V^{r\times r} = 2^{\frac{r(r-1)}{2}} \det \mathcal{M}_V^{r\times r}.$$

The result follows now by taking into account the result $\det \mathcal{M}_{CC}^{r\times r} = 1$, obained in [2]. □

An analogue proof can be done for the determinant of the matrix associated with the triangle $\mathcal{T}_{EV}$ written in its rectangular form.

Theorem 8. *Let $\mathcal{M}_{EV}^{r\times r}$ be the $r \times r$ matrix formed by the first r rows and r columns of a triangle $\mathcal{T}_{EV}$ written in its rectangular form. Then,*

$$\det \mathcal{M}_{EV}^{r\times r} = 2^{-\frac{r(r-1)}{2}}.$$

Proof. We consider now the sequence of the central binomial coefficients $(v_n)_n$ and the relation (2). Then,

$$v_n^{(k)} = 4^n\, c_n^{(k)}, \quad k = 0, 1, 2, \ldots, \; n = 0, 1, 2, \ldots.$$

If $\mathcal{M}_C^{r\times r}$ denotes the matrix formed by the first r rows and r columns of the convolution triangle of the sequence $(u_n^{(k-1)})_n$, then

$$\det \mathcal{M}_C^{r\times r} = 4^{\frac{r(r-1)}{2}} \det \mathcal{M}_{EV}^{r\times r}.$$

The result follows from $\det \mathcal{M}_C^{r\times r} = 2^{\frac{r(r-1)}{2}}$, computed in [2]. □

Observing Table 2, it stands out a pattern: for each m fixed, the odd $k = 2m-1$-th convolution of the even order-term sequence $(c_{2n})_n$ coincides with the sequence $(b_n)_n$ of the binomials

$$b_n = \binom{n+m-1}{m-1}, \; n = 0, 1, \ldots, \; m = 1, 2, \ldots, n,$$

generated by the functions (5). In other words, the triangle formed by the even columns of $\mathcal{T}_{EV}$ corresponds to the Pascal triangle. In the particular case of $m = 1$, (i.e., $k = 1$, the first convolution of $(c_{2n})_n$ by itself) we find the convolution formula

$$c_{2n}^{(1)} = \sum_{s=0}^{n} c_{2s} c_{2(n-s)} = \binom{n}{0} = 1,$$

which is just a rewrite of the well-known identity for convolution of the central binomial coefficients:

$$\sum_{s=0}^{n} \binom{2s}{s} \binom{2(n-s)}{n-s} = 4^n.$$

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Anti-Gaussian quadrature rules related to polynomials orthogonal on the unit semicircle

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Let Γ be a unit semicircle $\Gamma = \{z = e^{i\theta} : 0 \le \theta \le \pi\}$. Orthogonal polynomials on the unit semicircle with respect to the complex-valued inner product

$$\langle f, g \rangle = \int_\Gamma f(z)g(z)(iz)^{-1}dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})d\theta$$

was introduced by Gautschi and Milovanović in [1], were the certain basic properties were proved. Such orthogonality as well as the corresponding quadrature rules of Gaussian type were further studied in [2] and [4]. In this article we introduce anti-Gaussian quadrature rules related to the polynomials orthogonal on the unit semicircle (see [3]) and present stable numerical method for their construction. Some numerical examples are included, too.

2020 MSC: 65D32, 30C10

KEYWORDS: Complex-valued inner product, Orthogonal polynomials, Anti–Gaussian quadrature rule

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Bendersky-Adamchik constants, hyperfactorial functions, and Ramanujan summation of divergent series

Marc-Antoine Coppo

Introduced in 1933 by Bendersky, the sequence of mathematical constants called Bendersky-Adamchik constants (or generalized Glais-her–Kinkelin constants) arises quite naturally in number theory and analysis, and have been the subject of recent work. These constants are a natural extension of the classical Glaisher-Kinkelin constant

$$A = 1.282427\ldots,$$

and are closely related to the derivatives of the Riemann zeta function at negative integers. In this talk, we will focus on the connection between the logarithm of these constants and the sum of certain divergent series in the sense of Ramanujan's summation method.

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Keywords: Generalized Glaisher-Kinkelin constants, Generalized gamma functions, Ramanujan summation of series

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On the approximation to fractional calculus operators with multivariate Mittag-Leffler function in the kernel

Mehmet Ali Ozarslan

Many numerical techniques have been developed to approximate Riemann-Liouville and Caputo fractional calculus operators in various function spaces. Recently linear positive operators have been started to use to approximate fractional calculus operators such as Riemann-Liouville, Caputo, Prabhakar and operators containing bivariate Mittag-Leffler functions in the kernel. In the present paper, we first define and investigate the fractional calculus properties of Caputo derivative operator containing the multivariate Mittag-Leffler function in the kernel. Then we introduce approximating operators by using the modified Kantorovich operators to approximate fractional integral and Caputo derivative operators with multivariate Mittag-Leffler function in the kernel. We study the convergence properties of the operators and compute the degree of approximation by means of modulus of continuity and Holder continuous functions.

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KEYWORDS: Fractional calculus, Multivariate Mittag-Leffler function, Bernstein-Kantorovich operators, Laplace transform, Modulus of continuity

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Weakly analytic sets in locally convex spaces

Salvador López-Alfonso [1], *Manuel López-Pellicer* [*2], *and Santiago Moll-López* [3]

We characterize analytic sets Y in the weak topology $\sigma(E, E')$ of any locally convex space E in a large class $\mathfrak{G}$ of locally convex spaces that includes for instance (DF)-spaces and (LF)-spaces. We provide a general result (Theorem 5) which applies to show:

(i) The space $C(X)$ of continuous real-valued functions on a completely regular Hausdorff space X endowed with a topology ξ stronger or equal than the pointwise topology τ_p of $C(X)$ is analytic iff $(C(X), \xi)$ is separable and is covered by a directed family of ξ-compact sets, so-called compact resolution.

(ii) For a locally convex space E in class $\mathfrak{G}$ a subset Y of E is $\sigma(E, E')$-analytic iff Y has a $\sigma(E, E')$-compact resolution and is contained in a $\sigma(E, E')$-separable subset.

The later result easily applies to reprove a known important fact (due to Cascales and Orihuela) about weak metrizability of weakly compact sets in spaces in class $\mathfrak{G}$.

2020 MSC: 46A50, 46E10, 54H05

KEYWORDS: Analytic space, Compact resolution, $C_p(X)$-spaces, Locally convex spaces, Weak metrizability

Introduction

In the theory of locally convex spaces (lcs) E, working with compact sets in E two essential questions may arise, namely, about metrizability of such sets and (or) weakly angelicity of E. We refer to [4] (and references therein) providing a list of positive results (among others for (LF)-spaces and (DF)-spaces) concerning both questions. We recall only here that [4], being inspired by particular results for (LF)-spaces and (DF)-spaces and the common topological structure of the topological dual of each space of this type, introduced and studied a class $\mathfrak{G}$ of lcs for which both problems mentioned above have positive answers.

The class $\mathfrak{G}$ is indeed large and contains "almost all" important classes of locally convex spaces (including (LF)-spaces and (DF)-spaces), is stable by taking subspaces, Hausdorff quotients, countable direct sums, and products. Nevertheless, as we proved in [5], the space $C_p(X)$ of continuous real-valued maps on a Tychonoff space, i.e., a completely regular Hausdorff space X, endowed with the pointwise topology belongs to class $\mathfrak{G}$ iff $C_p(X)$ is metrizable. An interesting result from [4] states that a compact set K is Talagrand compact iff it is homeomorphic to a subset of a lcs in class $\mathfrak{G}$. Therefore, dealing with Talagrand compact sets one may ask when (weakly) compact sets in a lcs in class $\mathfrak{G}$ are (weakly) metrizable. Both questions were answered in [6] and [4], respectively, see also [11] (and references there).

We prove Theorem 5 (below) for spaces $C_p(X)$ and we applied it to show the following general

Theorem 1. *A subset Y of a lcs E in class $\mathfrak{G}$ is $\sigma(E,E')$-analytic iff Y has a $\sigma(E,E')$-compact resolution and is contained in a $\sigma(E,E')$-separable subset.*

Consequently, a lcs E in class $\mathfrak{G}$ is weakly analytic iff E is separable and admits a $\sigma(E,E')$-compact resolution. Note that the latter condition is equivalent to say that E is weakly K-analytic (since E is angelic by [4, Theorem 11] and we apply [3, Corollary 1.1]).

Although spaces $C_p(X)$ of continuous real-valued maps on X endowed with the pointwise topology τ_p do not belong to class $\mathfrak{G}$ for uncountable spaces X (as we have mentioned above), the argument used in the proof of Theorem 5 applies to characterize analytic spaces $C_p(X)$ for any completely regular Hausdorff space X. We prove that $C_p(X)$ is analytic iff $C_p(X)$ has a compact resolution and is separable, see Corollary 6.

Since every analytic compact set is metrizable [4, Theorem 15], Theorem 1 yields the following results from [6] and [4].

Corollary 2 (Cascales-Orrihuela)**.** *A $\sigma(E,E')$-compact set Y in a lcs E in class $\mathfrak{G}$ is $\sigma(E,E')$-metrizable iff Y is contained in a $\sigma(E,E')$-separable subset of E.*

Theorem 3 (Cascales-Orihuela)**.** *A precompact set Y in a lcs in class $\mathfrak{G}$ is metrizable.*

A family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of sets covering a set X is called a *resolution* of X if $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. A lcs E belongs to class $\mathfrak{G}$ if there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ in $(E', \sigma(E', E))$ (called a $\mathfrak{G}$-representation of E') such that each sequence in any A_α is equicontinuous, [4].

A topological space X is *web-bounding* [12, Note 3] if there is a family $\{A_\alpha : \alpha \in \Omega\}$ of subsets of X for some non-empty $\Omega \subset \mathbb{N}^\mathbb{N}$ whose union X_0 is dense in X, called a web-bounding representation, such that if $\alpha = (n_k) \in \Omega$ and $x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Omega, m_j = n_j, j = 1,\ldots,k\}$, then $(x_k)_k$ is functionally bounded. If the same holds for $X = X_0$, we call X *strongly web-bounding.*

A Tychonoff space X is called a *Lindelöf Σ-space* [1] (or K-countably determined [14]) if there is an upper semi-continuous compact-valued map from a non-empty subset $\Omega \subset \mathbb{N}^\mathbb{N}$ covering X. If the same holds for $\Omega = \mathbb{N}^\mathbb{N}$, then X is called *K-analytic.* X is *quasi-Suslin* if there exists a set-valued map T from $\mathbb{N}^\mathbb{N}$ into X covering X which is quasi-Suslin, i.e., if $\alpha_n \to \alpha$ in $\mathbb{N}^\mathbb{N}$ and $x_n \in T(\alpha_n)$, then $(x_n)_n$ has a cluster point in $T(\alpha)$, see [17].

Note that K-analytic $\Leftrightarrow$ Lindelöf $\wedge$ quasi-Suslin, and K-analytic $\Rightarrow$ Lindelöf Σ. A topological space X is *analytic* if it is a continuous image of the space $\mathbb{N}^\mathbb{N}$. Every K-analytic space has a compact resolution, see [15], or [3], and every angelic space with a compact resolution is K-analytic, see [3, Corollary 1.1].

Lemmas and Theorem 1 proof

We need the following result [15].

Proposition 4 (Talagrand)**.** *Let (X,ξ) be a regular space which admits a stronger topology ϑ such that (X,ϑ) is a Lindelöf Σ-space. Then $d(X,\vartheta) \leq \omega(X,\xi)$, where $d(X)$ and $\omega(X)$ denote the density and the weight of X, respectively.*

Recall, that, for example, topological spaces containing dense quasi-Suslin spaces are web-bounding, [12]. Hence every space containing a dense σ-compact space is web-bounding, in particular separable spaces are web-bounding. On the other hand, a metrizable space is web-bounding iff it is separable, apply [4, Theorem 1, Note 4]. The following result uses some ideas from [4].

Theorem 5. *Let X be a web-bounding space. A non-empty set $Y \subset C_p(X)$ is analytic iff Y has a compact resolution and is contained in a separable subset of $C_p(X)$.*

In [16] Tkachuk proved that $C_p(X)$ is K-analytic iff it has a compact resolution. If X is a separable metric space, then $C_p(X)$ is analytic iff it admits a resolution consisting of bounded sets, see [2, Corollary 2.5] and [7, Proposition 1]. We have the following variant for analyticity of $C_p(X)$ for arbitrary X.

Corollary 6. *Let ξ be a topology on $C(X)$ which is stronger or equal than the pointwise topology τ_p of $C(X)$. Then $(C(X), \xi)$ is analytic iff $(C(X), \xi)$ is separable and has a ξ-compact resolution.*

Consequently, for separable spaces $C_p(X)$ if there is an upper semi-continuous compact-valued map from $\mathbb{N}^{\mathbb{N}}$ covering $C_p(X)$, then $C_p(X)$ is a continuous image of $\mathbb{N}^{\mathbb{N}}$.

Example 7. *Corollary 6 fails for the weak*-dual $L_p(X)$ of $C_p(X)$.*

Proof. Let $X := [0,1]^{\mathbb{R}}$ be endowed with the product topology. Then X is K-analytic separable but not analytic. Consequently $L_p(X)$ is K-analytic. Further, since X is separable, so does $L_p(X)$ by [1, Proposition 0.5.14]. It is not analytic, since X in $L_p(X)$ is closed and each closed subspace of an analytic space is analytic. □

We finish this abstract with the prove of Theorem 1.

Proof. Note that $(E', \sigma(E', E))$ is strongly web-bounding. Indeed, let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E'. If $x_k \in C_{n_1,n_2,\ldots,n_k}$ for all $k \in \mathbb{N}$, there exists $\beta_k \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \in A_{\beta_k}$. Since there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $\beta_k \leq \gamma$, we have $x_k \in A_\gamma$ for all $k \in \mathbb{N}$ yielding equicontinuity of $(x_k)_k$, so $(x_k)_k$ is functionally bounded. As $(E, \sigma(E, E'))$ is contained in $C_p(E', \sigma(E', E))$ we apply Theorem 5. □

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Discrete time mean-field optimal control problems: Stochastic maximum principle

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We study optimal control of a discrete-time stochastic differential equation of mean-field type with coefficients dependent on function of the law and state of the process. A new version of the maximum principle for discrete-time mean-field type stochastic optimal control problems is presented, using new discrete-time mean-field backward stochastic equations. The cost functional is also of mean-field type. The study derives necessary first-order and sufficient optimality conditions using adjoint equations that take the form of discrete-time backward stochastic differential equations with a mean-field component.

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Keywords: Discrete-time backward stochastic equation, Mean-field theory, Necessary and sufficient conditions, Optimal control problem, Stochastic maximum principle

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Generating functions for superoscillating functions and recurrence relations

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In this contribution we study particular generating functions for the coefficients of the classical superoscillatory functions. On the basis of the concept of generating functions we are able to develop some new explicit links between the superoscillatory coefficients and important families of special polynomials, numbers, and special functions. In particular we obtain relations with the generalized Hermite polynomials, Bernstein functions and the Stirling partition numbers of second kind. The concept of generating functions furthermore allows us to develop an explicit recurrence relation and a derivative formula for the superoscillatory coefficients.

2020 MSC: 33C15, 32A15, 47B38

KEYWORDS: Superoscillations, Generating functions, Stirling numbers, Bernstein basis functions, Generalized Hermite polynomials, Recurrence relations

The Aharonov-Berry superoscillating functions and their associated coefficients

This paper deals with band-limited functions that can oscillate faster than their fastest Fourier component. Such functions are called superoscillations. A standard reference for superoscillations is for instance [2]; and an abundant reference list is provided also in our paper [4] where we discuss the topic of this presentation in all detail.

The most basic prototype of superoscillations is given by

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \sum_{k=0}^{n} c_k(n,a)e^{i(1-2k/n)x}, \tag{1}$$

where $x \in \mathbb{R}$ and, for $a > 1$, the coefficients $c_k(n,a)$ are given by

$$c_k(n,a) = \binom{n}{k}\left(\frac{1+a}{2}\right)^{n-k}\left(\frac{1-a}{2}\right)^{k}. \tag{2}$$

For a fixed $x \in \mathbb{R}$ we obtain when considering $n \to \infty$:

$$\lim_{n\to\infty} F_n(x,a) = e^{iax}.$$

Note that in the Fourier representation of F_n we have that the exponent $|1 - 2k/n| < 1$, however the limit function e^{iax} oscillates with frequency a which can be arbitrarily large. This is the so-called superoscillatory phenomenon.

The aim of this presentation is to study the generating functions associated with the sequence $c_k(n,a)$ defined in (2).

To start let us take a sequence $(a_n)_{n\in\mathbb{N}_0}$ and define as usual a generating function as the formal power series by

$$f(t) = \sum_{n=0}^{\infty} a_n t^n.$$

It is important to note that there are various types of generating functions, including for example exponential generating functions, Lambert series, Bell series, Dirichlet series and many more, cf. for example [9].

The exponential generating function of a sequence is defined by

$$EG(a_n;t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}.$$

If $EG(a_n;t)$ is analytic in an open set containing 0, then a_n can be explicitly obtained as the sequence of the derivatives with respect to t of the function $EG(a_n,\cdot)$ evaluated at 0.

The plan of the presentation and the main results.

In *Section 2* we introduce the generating functions for superoscillations. We express them by two functions S_1 and S_2 which in turn exhibit deep connections with the coefficients $c_k(n,a)$.

The functions S_1 and S_2 depend on some parameters m, k, n and on the auxiliary variables $(\alpha_0,\alpha_1,\dots,\alpha_m)$. For particular values on m we have remarkable relations with the coefficients $c_k(n,a)$, in particular explicit relations that involve the Stirling partition numbers of second kind.

One of our aim consists in understanding more profoundly why Stirling numbers of the second kind play a key role in superoscillatory functions.

In *Section 3* we then turn to study recurrence relations for $c_k(n,a)$ and relate them to special polynomials, in particular to Bernstein basis functions $B_k^v(x)$ and generalized Hermite polynomials.

Generating functions for superoscillations

We start with fixing some notation.

Following for example [7], let $\beta\in\mathbb{C}$ and $v\in\mathbb{N}_0$. Write the Pochhammer symbol as

$$(\beta)^{\overline{v}} = \prod_{j=0}^{v-1}(\beta+j),$$

where $(\beta)^{\overline{0}} = 1$ for $\beta\neq 1$. For $p,q\in\mathbb{N}$ the generalized hypergeometric function is

$${}_pF_q\left[\begin{array}{c}\beta_1,\beta_2,\dots,\beta_p\\ \gamma_1,\gamma_2,\dots,\gamma_q\end{array};z\right] = \sum_{m=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(\beta_j)^{\overline{m}}}{\prod_{j=1}^{q}(\gamma_j)^{\overline{m}}}\right)\frac{z^m}{m!}.$$

This series convergences for all z if $p<q+1$, and for $|z|<1$ if $p=q+1$.

It is possible to consider $\beta_j\in\mathbb{C}$, $\gamma_j\in\mathbb{C}\setminus-\mathbb{N}_0=\mathbb{C}\setminus\{0,-1,-2,\cdots\}$. Additionally, one puts ${}_0F_0(z)=e^z$.

Next, in line with our paper [4], we introduce the functions S_1 and S_2 which will be of fundamental importance for the representation of the coefficients of the superoscillatory functions F_n:

Definition 1 (The functions S_1 and S_2). *Let $p, q \in \mathbb{N}$, $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$, $x, t \in \mathbb{R}$ and $k, m, n \in \mathbb{N}$.*

Put

$$\beta_1 = \beta_2 = \cdots = \beta_m = k,$$

and

$$\gamma_1 = \gamma_2 = \cdots = \gamma_m = k + 1.$$

Then we define the function S_1 by

$$\begin{aligned} & S_1\left(t, x; m, k, n; \alpha_0, \alpha_1, \ldots, \alpha_m\right) \\ := \quad & \frac{1}{k!}\left(\frac{1-x}{2}t\right)^k \sum_{j=0}^{m} \alpha_j \sum_{l=0}^{j} \binom{j}{l} \left(-\frac{2k}{n}\right)^{j-l} \\ \times \quad & {}_lF_l\left[\begin{matrix} k, k, ..., k \\ k+1, k+1, ..., k+1 \end{matrix}; \frac{1+x}{2}t\right]. \end{aligned} \tag{3}$$

Next, put

$$\beta_1 = \beta_2 = ... = \beta_m = k + 1,$$

and

$$\gamma_1 = \gamma_2 = ... = \gamma_m = k.$$

Then we define the function S_2 by

$$\begin{aligned} & S_2\left(t, x; m, k, n; \alpha_0, \alpha_1, \ldots, \alpha_m\right) \\ := \quad & \frac{1}{k!}\left(\frac{1-x}{2}t\right)^k \sum_{j=0}^{m} \alpha_j \sum_{l=0}^{j} \binom{j}{l} \left(-\frac{2k}{n}\right)^{j-l} \\ \times \quad & {}_lF_l\left[\begin{matrix} k+1, k+1, ..., k+1 \\ k, k, ..., k \end{matrix}; \frac{1+x}{2}t\right]. \end{aligned} \tag{4}$$

Now S_1 and S_2 generate the following sequences b_1 and b_2:

Definition 2 (The sequences b_1 and b_2). *Let $x > 1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, v$. Let $\alpha_0, \ldots, \alpha_m \in \mathbb{R}$. The sequences b_1 and b_2 are implicitly defined as coefficient functions in the representations*

$$S_1\left(t, x; m, k, n; \alpha_0, \alpha_1, \ldots, \alpha_m\right) = \sum_{v=0}^{\infty} b_1\left(v, x; m, k, n; \alpha_0, \alpha_1, \ldots, \alpha_m\right) \frac{t^v}{v!}, \tag{5}$$

and

$$S_2\left(t, x; m, k, n; \alpha_0, \alpha_1, \ldots, \alpha_m\right) = \sum_{v=0}^{\infty} b_2\left(v, x; m, k, n; \alpha_0, \alpha_1, \ldots, \alpha_m\right) \frac{t^v}{v!}. \tag{6}$$

Finally, we introduce a generating function for the superoscillatory coeffients $c_k(n, a)$.

Definition 3 (Generating function of the coefficients c_k). *Let $c_k(n, x)$ be the coefficients of the superoscillating function F_n. We define its (exponential) generating function by*

$$G_k(t, x) := \sum_{v=0}^{\infty} c_k(v, x) \frac{t^v}{v!}. \tag{7}$$

If one inserts $m = 0$ into (3) and (4) and if one further considers the particular case $\alpha_0 = 1$, then one can deduce the following formula

Lemma 4. *For every $t \in \mathbb{R}$ and $x \in \mathbb{R}$ we have*

$$G_k(t,x) = \frac{1}{k!}\left(\frac{t(1-x)}{2}\right)^k e^{t\left(\frac{x+1}{2}\right)} = \sum_{v=0}^{\infty} c_k(v,x)\frac{t^v}{v!}. \tag{8}$$

For the detailed proof we refer the reader to [4].

Now on the basis of this formulae one can find an explicit connection between the function $S_2(t,x;1,m,n;\alpha_0,\alpha_1,\ldots,\alpha_m)$ and the Stirling partition numbers of the second kind $\mathcal{S}_2(c,d)$:

Theorem 5. *Let*

$$\mathcal{S}_2(c,d) = \frac{1}{d!}\sum_{v=0}^{d}\left(\begin{array}{c} d \\ v \end{array}\right)(-1)^v(d-v)^c$$

be the Stirling numbers of the second kind. Then we have

$$\begin{aligned} & S_2(t,x;1,m,n;\alpha_0,\alpha_1,\ldots,\alpha_m) \\ = \; & \frac{1}{k!}\left(\frac{1-x}{2}t\right)^k \sum_{j=0}^{m}\alpha_j \sum_{l=0}^{j}\left(\begin{array}{c} j \\ l \end{array}\right)\left(-\frac{2k}{n}\right)^{j-l} \\ & \times e^{\left(\frac{1+x}{2}t\right)}\sum_{c=0}^{l}\mathcal{S}_2(l+1,c+1)\left(\frac{1+x}{2}t\right)^c. \end{aligned} \tag{9}$$

Also a relation between the sequence $b_2(v,x;m,k,n;\alpha_0,\alpha_1,\ldots,\alpha_m)$, the coefficients $c_k(n,a)$, and the Stirling numbers $\mathcal{S}_2(c,d)$ can be established. We can prove

Theorem 6. *Let $x > 1$, $m, v \in \mathbb{N}_0$, and $n \in \mathbb{N}$ and $k = 0,1,2,\ldots,v$. Let $\alpha_1,\ldots,\alpha_m$ be real numbers. Then we have*

$$\begin{aligned} & b_2(v,x;m,k,n;\alpha_0,\alpha_1,\ldots,\alpha_m) \\ = \; & \sum_{j=0}^{m}\alpha_j\sum_{l=0}^{j}\left(\begin{array}{c} j \\ l \end{array}\right)\left(-\frac{2k}{n}\right)^{j-l}\sum_{c=0}^{l}\left(\begin{array}{c} l \\ c \end{array}\right) \\ & \times\sum_{d=0}^{c}\left(\begin{array}{c} v \\ d \end{array}\right)\frac{d!\mathcal{S}_2(c,d)}{k^c}\left(\frac{1+x}{2}\right)^d c_k(v-d,x). \end{aligned} \tag{10}$$

Remark 1. *Putting $m = 0$ in (10) together with $\mathcal{S}_2(0,0) = 1$ produces an explicit relation between b_2 and the coefficients c_k:*

$$b_2(v,x;0,k,n;\alpha_0) = \alpha_0 c_k(v,x).$$

Also here we refer the interested reader for the detailed proofs and further details and relation to our new paper [4]. See also the recent paper [8] where also Stirling numbers have been related to superoscillatory problems behind a more physical background and from a different perspective.

A recurrence relation for the coefficients $c_k(n,x)$ and their relation to special polynomials

As shown in detail in [4] one can express $c_k(v,x)$ in terms of the so-called Bernstein basis functions $B_k^v(x)$ which are generated by

$$\frac{1}{k!}(ty)^k e^{t(1-y)} = \sum_{v=0}^{\infty} B_k^v(x)\frac{t^v}{v!},$$

cf. [5, 11]. Following [4] we can establish the relation

$$c_k(n,y) = B_k^n\left(\frac{1-y}{2}\right).$$

This allows us to establish the following

Lemma 7 (Derivative formula)**.** *Let $k,n \in \mathbb{N}$. Then*

$$\frac{d}{dx}c_k(n,x) = \frac{n}{2}\left(c_k(n-1,x) - c_{k-1}(n-1,x)\right).$$

For the proof we again refer to [4].

This tool in hand permits us to prove the following main result of this section:

Theorem 8 (Recurrence relation for $c_k(n,x)$)**.** *Let $x > 1$. Let $n \in \mathbb{N}_0$ and $k = 1,2,\ldots,n$. We have*

$$c_k(n+1,x) = \frac{1-x}{2}c_{k-1}(n,x) + \frac{1+x}{2}c_k(n,x).$$

Proof. By combining the following equation

$$\frac{\partial}{\partial t}G_k(t,x) = \frac{1-x}{2}G_{k-1}(t,x) + \frac{1+x}{2}G_k(t,x)$$

considering the partial derivation with respect to t with equation (8), we obtain a recurrence relation for the $c_k(n,x)$.

$$\sum_{n=0}^{\infty} c_k(n+1,x)\frac{t^n}{n!} = \frac{1-x}{2}\sum_{n=0}^{\infty} c_{k-1}(n,x)\frac{t^n}{n!} + \frac{1+x}{2}\sum_{n=0}^{\infty} c_k(n,x)\frac{t^n}{n!}.$$

A comparison of the coefficients of $\frac{t^n}{n!}$ on both sides leads to the statement. □

Finally let us present a link between $c_k(n,a)$ and the famous Hermite polynomials.

The generalized Hermite polynomials, also called the Gould-Hopper polynomials $H_n^{(j)}(x,y)$, cf. [6] are implicitly defined by the generating function

$$G_H(t,x,y,j) = \exp\left(xt + yt^j\right) = \sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!}. \quad (11)$$

The polynomials $H_n^{(2)}(x,y)$ satisfy the following heat equation:

$$\frac{\partial}{\partial y}\left(H^{(2)}(x,y)\right) = \frac{\partial^2}{\partial x^2}\left(H^{(2)}(x,y)\right),$$

cf. [3].

Substituting $y=-1$, $j=2$ and $x=2z$ into (11), we re-obtain the generating function for the classical Hermite polynomials, $H_n(z) := H_n^{(2)}(2z,-1)$:

$$F_H(t,z) = e^{2zt-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}.$$

From the previous equation, one may infer that

$$H_m(z) = (-1)^m e^{z^2}\frac{d^m}{dz^m}\left(e^{-z^2}\right),$$

$m \in \mathbb{N}_0$, and also

$$H_n(z) = n!\sum_{s=0}^{\left[\frac{n}{2}\right]}(-1)^s\frac{(2x)^{n-2s}}{(n-2s)!s!}$$

where $[x]$ denote the largest integer $\leq x$.

Finally, using representations of hypergeometric functions and applying their functional equation (see also [10]) allows us to also set up an explicit link between the Hermite polynomials $H_n(x,y)$ and the superoscillatory coefficient functions $c_k(n,x)$.

Theorem 9 (*cf.* [4])**.** *Let $x>1$. Let $n \in \mathbb{N}_0$ and $k=0,1,2,\ldots,n$. We have*

$$\left(\frac{1-x}{2}\right)^k H_{n-k}\left(\frac{1+x}{4}\right) = \frac{n!}{\binom{n}{k}}\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^j\, c_k(n-2j,x)}{j!(n-2j)!}.$$

where $[x]$ denotes the largest integer $\leq x$.

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S-versions of some module theoretic concepts

Secil Ceken

In recent years, some ring and module theoretic concepts are generalized via a multiplicatively closed subset S of a commutative ring with identity R. The most important ones of these concepts are S-prime submodules, S-second submodules and their generalizations. In this talk, we introduce and investigate the concept of S-semisecond submodules as a generalization of semisecond and S-second submodules. We also give some results on S-semiprime submodules and investigate some interrelations between S-semiprime and S-semisecond submodules. In addition, to give many examples and characterizations of S-semisecond submodules, we characterize a certain class of semisecond submodules in terms of S-semisecond submodules. We also determine several characterizations of modules M in which every submodule N of M with $ann_R(N) \cap S = \emptyset$ is S-semisecond.

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KEYWORDS: S-prime submodule, S-semiprime submodule, S-second submodule, S-semisecond submodule

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Inference for first passage times of the Feller process

Satish Iyengar [*1] *and Bowen Yi* [2]

The Feller diffusion process has linear drift and a state-dependent diffusion coefficient that vanishes at zero. Earlier studies have shown that it provides a better fit for neural activity than the Ornstein-Uhlenbeck under certain conditions. In this talk we describe inference based on maximum likelihood for this model when the available data are spike trains rather than the neuron's subthreshold voltage traces.

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Keywords: Confluent hypergeometric function, Cox-Ingersoll-Ross model, Laplace transform inversion, Maximum likelihood, Tauberian theorem

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Umbral convolutions of hypergeometric functions

Subuhi Khan

The employment of umbral approach has been demonstrated to be a useful tool for dealing with the unified handling of special and non-special functions [3]-[7]. The methods associated with umbral approach have been widely employed to investigate the characteristics of special functions. The umbral approach is a versatile instrument for explicit calculations that aids in the development of concepts in the setting of operational calculus.

The philosophy that underpins umbral formalism is summarised as follows: a collection of operators is designed to introduce the image function of another function with unknown attributes, known as the object function. The image function is chosen in such a way that the relevant (known) properties are used to determine the counterpart attributes of the object function. The aforementioned objective is achieved by employing the principle of permanence of formal properties proposed in [7].

The hypergeometric functions [1] are unique functions, which are considered as generalizations of higher-order transcendental functions. These functions are used in variety of fields, such as number theory, theory of Eulerian differential operators, theoretical physics and are also explored extensively in the theory of statistical and economic distributions. It may be observed that the hypergeometric function encompasses the majority of functions examined by physicists, engineers, economists and mathematicians.

The umbral restyling of the hypergeometric functions is shown to be a useful and efficient approach in simplifying the associated computational technicalities. Very recently, Dattoli, M. Haneef, Subuhi Khan and Licciardi [2] applied the umbral formalism to the Gauss hypergeometric functions and their relevant generalizations. It is shown that the suggested approach is particularly efficient for evaluating integrals involving hypergeometric functions and their combinations with other special functions.

This work aims to show that the umbral methods can be used to convolute special functions in order to introduce new families of special functions. It is important to observe that it is not possible to study these hybrid families of special functions by using ordinary operational methods, as for example, monomiality principle [8]. In this work, the umbral convolutions of Bessel and Tricomi functions with hypergeometric functions are considered.

As a result, two novel extensions of hypergeometric functions namely the hypergeometric-Bessel and hypergeo- metric-Tricomi functions are introduced. The methods of umbraloperational nature are used to establish the generating function, explicit representation, differential recurrence relations, integral representations and other properties of these functions.

In view of the fact that while converting cylinder and plain waves or visualising frequency modulation signals, the Jacobi-Anger expansion formulae play a crucial role in physics and signal processing. A plane wave is depicted in cylindrical coordinates by use of these expansions. In view of the importance of these expansions, the Jacobi-Anger expansion formulae for hypergeometric-Bessel functions ${}_{2\mathcal{F}_1}J_m\,(a,b;c;x)$ are obtained.

In science, engineering, technology, finance, and other fields, graphs are useful tools for a variety of tasks. Various functions and other qualitative structures are displayed, examined, clarified, and interpreted using graphical representations. The graphs for hypergeometric-Bessel ${}_2\mathcal{F}_1 J_m\,(a,b;c;x)$ and hypergeometric-Tricomi functions ${}_2\mathcal{F}_1 C_m\,(a,b;c;x)$ for different values of m using the Mathematica software are presented. The pattern of distribution of zeros of these functions, for some particular values of their parameters and indices are also examined.

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Identities on degenerate hyperharmonic numbers

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Recent explorations for various degenerate versions of quite a few special numbers and polynomials have been fascinating and fruitful, which began with the pioneering work of Carlitz in [2, 3]. These quests for degenerate versions are not only limited to special polynomials and numbers but also extended to some transcendental functions, like gamma functions (see [13]). Many different tools are used, which include generating functions, combinatorial methods, p-adic analysis, umbral calculus, operator theory, differen- tial equations, special functions, probability theory and analytic number theory (see [8, 10-12, 15, 17] and the references therein). It is also worth mentioning that λ-umbral calculus has been introduced in [11], which turns out to be more convenient than the 'classical' umbral calculus when dealing with degenerate Sheffer polynomials.

The aim of this talk is to investigate some properties, recurrence relations and identities involving degenerate hyperharmonic numbers, hyperharmonic numbers and degenerate harmonic numbers (see (1.3), (1.4), (1.6)). The novelty of this talk is the derivation of an explicit expression of the degenerate hyperharmonic numbers in terms of the degenerate harmonic numbers (see Theorem 2.3). This is a degenerate version of the corresponding identity representing the hyperharmonic numbers in terms of harmonic numbers due to Conway and Guy (see (1.5)). The outline of this talk is as follows. In Section 1, we recall the degenerate exponentials and the degen- erate logarithms. We remind the reader of the harmonic numbers and their generating function, and of the degenerate harmonic numbers and their generating function. Then we recall the hyperharmonic numbers due to Conway and Guy [5], its explicit expression in terms of harmonic numbers and their generating func- tion. Finally, we remind the reader of the recently introduced degenerate hyperharmonic numbers, which are a degenerate version of the hyperharmonic numbers, and of their generating function. Section 2 is the main result of this talk. We obtain an identity involving the degenerate hyperharmonic numbers and the hyper harmonic numbers in Theorem 2.2. It is obtained by taking higher order derivatives of the generating function of the degenerate hyperharmonic numbers in (1.7). Theorem 2.3 is a degenerate version of the explicit expres- sion for the hyperharmonic numbers (1.5), which is obtained from Theorem 2.2 and Lemma 2.1 about explicit expressions of certain polynomials. In Section 3, we get an identity involving the degenerate hyperharmonic numbers and the degenerate zeta function.

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KEYWORDS: Degenerate hyperharmonic number, Degenerate harmonic number, Hyperharmonic number, Degenerate Hurwitz zeta function

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Focus on connectivity indices and its impact

Veerebradiah Lokesha

Last two decades connectivity indices as grown in the field of discrete structures as well as application in multidisciplinary aspects. More than 8 thousand new connectivity indices are introduced and many inter connected fields research activities are in full swing. The term connectivity index is often reserved for graph invariant in molecular graph theory. In the Mathematical and Chemical literature, plenty of connectivity indices have been introduced and extensively studied. The concept of VL index was recently introduced by Deepika T in the Chemical graph theory. It is a degree based connectivity indices. In 2021, motivated from this Suvarna and et al. introduced the VL temperature index and status index for graph structures. These two indices are well correlated to butane structure. Connectvity indices are used for quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) studies.

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2 CONTRIBUTIONS

On the relations derived from consideration of generating functions for sum of higher powers of inverse binomial coefficients

Yilmaz Simsek

In this presentation, we survey certain families of the sums of higher powers of binomial coefficients and higher powers of inverse binomial coefficients. We investigate some properties of these sums with the aid of hypergeometric series and certain family of special numbers and polynomials, involving the Apostol-Bernoulli numbers and polynomials, the Euler-Frobenius polynomials, B-spline, exponential Euler spline, the Narayana numbers, and the others. Finally, we give some useful open problems involving Eulerian statistic, sum of higher powers of inverse binomial coefficients, the Narayana polynomials, the Catalan numbers, the Eulerian spline.

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KEYWORDS: Generating functions, Catalan numbers, Narayana numbers and polynomials, Exponential Euler spline, Sums of powers of (inverse) binomial coefficients

Introduction

Ordinary or exponential generating functions have many vital applications in all branches of mathematics and other applied sciences, not only for finite sums and infinite sums involving special numbers and polynomials, but also for use in moment calculus, variance calculus, expected value calculus, solutions of differential equations, etc. In particular, generating function families for certain finite sums containing the powers of the (inverse) binomial coefficients, which is the subject of this study, are also used in mathematics and other applied sciences (*cf.* [1]-[44]).

In [32] and [37], we constructed generating functions for the finite sums of powers of (inverse) binomial coefficients. In these papers we know that the finite sums of powers of binomial coefficients have many applications in mathematical models, the Franel numbers, Brownian paths, graph theory, and other real world problems that are associated watermelons with more than two chains (*cf.* [1]- [43]). In [37], we show that a certain family of infinite sum of powers of inverse binomial coefficients were related to the Lerch zeta functions, the Euler splines and other special functions.

We [32] defined the following finite sums of powers of binomial coefficients, $y_6(j, b; \omega, p)$:

$$y_6(j, b; \omega, p) = \sum_{k=0}^{b} \binom{b}{k}^p \frac{\omega^k k^j}{b!}, \tag{1}$$

where $b, j, p \in \mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$), $\omega \in \mathbb{R}$, set of real numbers or $\omega \in \mathbb{C}$, set of complex numbers, and $0^j = 1$ for $j = 0$.

Generating function expressed in terms of generalized hypergeometric series for (1) is given by

$$\begin{aligned} F_{y_6}(u,b;\omega,p) &= {}_pF_{p-1}\left(-b,-b,...,-b;1,1,...,1;(-1)^p\,\omega e^u\right) \\ &= \sum_{j=0}^{\infty} b!y_6(j,b;\omega,p)\frac{u^j}{j!}, \end{aligned} \tag{2}$$

(*cf.* [32]), where ${}_kF_m\left(x_1,...,x_k;w_1,...,w_m;u\right)$ denotes the following generalized hypergeometric series:

$${}_kF_m\left(x_1,...,x_k;w_1,...,w_m;u\right) = \sum_{n=0}^{\infty}\left(\frac{\prod_{j=1}^{k}\left(x_j\right)_n}{\prod_{j=1}^{m}\left(w_j\right)_n}\right)\frac{u^n}{n!}, \tag{3}$$

where the above series converges for all u if $k < m+1$, and for $|u| < 1$ if $k = m+1$. All parameters have general values, real or complex, except for the w_j, $j = 1,2,...,m$ none of which is equal to zero or a negative integer. $(w)_v$ denotes the Pochhammer's symbol, defined by

$$(w)_v = \prod_{j=0}^{v-1}(w+j)$$

and $(w)_0 = 1$, for $w \neq 1$, $v \in \mathbb{N}$, and $w \in \mathbb{C}$ (*cf.* [10, 22, 32, 41]).

Here we note that using (1) and (2), we solved Srivastava's [40, p. 416] open problem 1.

Putting $k = p+1$, $m = p$ and $x_1 = x_2 = \cdots = x_{p+1} = 1$, $w_1 = w_2 = \cdots = w_p = -\beta$, $(-\beta \notin \{0,-1,-2,-3,\ldots\})$, and $u = (-1)^p\,\lambda e^z$ in (3), we [37] constructed the following generating function expressed in terms of generalized hypergeometric series for higher powers of inverse binomial coefficients:

Let v, $p \in \mathbb{N}_0$, $-\beta \notin \{0,-1,-2,-3,\ldots\}$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. We define the sum $B_v(\beta;\lambda,p)$ by the following generating function:

$$\begin{aligned} F(z,v;\beta,\lambda,p) &= {}_{p+1}F_p\left(1,1...,1;-\beta,...,-\beta;(-1)^p\,\lambda e^z\right) \\ &= \sum_{v=0}^{\infty} B_v(\beta;\lambda,p)\frac{z^v}{v!} \end{aligned} \tag{4}$$

and

$$\sum_{j=0}^{\infty}\frac{\left(\lambda e^z\right)^j}{\binom{\beta}{j}^p} = \sum_{v=0}^{\infty} B_v(\beta;\lambda,p)\frac{z^v}{v!}. \tag{5}$$

Using (4), we [37] gave the following some numerical values of the sum $B_v(\beta;\lambda,p)$:

$$\begin{aligned} B_1(\beta;\lambda,1) &= \frac{\lambda\ {}_2F_1(2,2;1-\beta;-\lambda)}{\beta}, \\ B_1(\beta;\lambda,2) &= \frac{\lambda\ {}_3F_2(2,2,2;1-\beta,1-\beta;\lambda)}{\beta^2}, \\ B_2(\beta;\lambda,1) &= \frac{\lambda\ {}_3F_2(2,2,2;1,1-\beta;-\lambda)}{\beta}, \\ B_2(\beta;\lambda,2) &= \frac{\lambda\ {}_4F_3(2,2,2,2;1,1-\beta,1-\beta;\lambda)}{\beta^2}, \end{aligned}$$

and also

$$B_3(\beta;\lambda,1)=\frac{\lambda\ {}_4F_3(2,2,2,2;1,1,1-\beta;-\lambda)}{\beta},$$
$$B_3(\beta;\lambda,2)=\frac{\lambda\ {}_5F_4(2,2,2,2,2;1,1,1-\beta,1-\beta;\lambda)}{\beta^2},$$
$$B_3(\beta;\lambda,3)=\frac{\lambda\ {}_6F_5(2,2,2,2,2,2;1,1,1-\beta,1-\beta,1-\beta;-\lambda)}{\beta^3},$$
$$B_3(\beta;\lambda,4)=\frac{\lambda\ {}_7F_6(2,2,2,2,2,2,2;1,1,1-\beta,1-\beta,1-\beta,1-\beta;\lambda)}{\beta^4},$$

and so on.

With the aid of (4), we [37] also gave

$$B_v(\beta;\lambda,p)=\sum_{m=0}^{\infty}\frac{m^v\lambda^m}{\binom{\beta}{m}^p} \tag{6}$$

and

$$B_v(\beta;\lambda,p)=\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}\sum_{k=0}^{j}\binom{j}{k}k!S(v,k),$$

where v, $p\in\mathbb{N}_0$ and $\lambda\in\mathbb{R}$ (or $\mathbb{C}$) with $|\lambda|<1$,

$$\binom{\beta}{0}=1$$

and

$$\binom{\beta}{m}m!=\prod_{j=0}^{m-1}(\beta-j).$$

We [37] gave

$$\mathcal{S}_v(n;\lambda,p):=\sum_{j=0}^{n}\frac{j^v}{\binom{n}{j}^p}\lambda^j, \tag{7}$$

where n, v, $p\in\mathbb{N}_0$ and $\lambda\in\mathbb{R}$ (or $\mathbb{C}$).

Putting $v=0$ and $\lambda=p=1$ in (7), we have

$$\mathcal{S}_0(n;1,1)=\sum_{m=0}^{n}\frac{1}{\binom{n}{m}}, \tag{8}$$

which have the following generating functions

$$\frac{\ln(1-u)}{\frac{1}{2}u^2-u}=\sum_{n=0}^{\infty}\mathcal{S}_0(n;1,1)u^n, \tag{9}$$

$$\frac{2}{2-3u+u^2}-\frac{2\ln(1-u)}{(2-u)^2}=\sum_{n=0}^{\infty}\mathcal{S}_0(n;1,1)u^n, \tag{10}$$

$$-\frac{2u\ln(1-u)}{(2-u)^3}-\frac{u\,(3u-4)}{(2-3u+u^2)^2}=\sum_{n=0}^{\infty}\mathcal{S}_1(n;1,1)u^n$$

(*cf.* [8, Exercise 30*, p. 272], [3, 20, 24, 37]).

We [37] defined the following two new class of polynomials

$$\mathcal{P}_v(x,\beta;\lambda,p) = \sum_{j=0}^{v}\binom{v}{j}B_j(\beta;\lambda,p)x^{v-j} = \sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}(j+x)^v, \tag{11}$$

which is come from the following generating function

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}e^{(j+x)z} = \sum_{v=0}^{\infty}\mathcal{P}_v(x,\beta;\lambda,p)\frac{z^v}{v!} \tag{12}$$

where $-\beta \notin \{0,-1,-2,-3,\ldots\}$, v, $p \in \mathbb{N}_0$ and $x,\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda|<1$ and

$$\mathcal{Q}_v(x,n;\lambda,p) = \sum_{j=0}^{v}\binom{v}{j}x^{v-j}\mathcal{S}_j(n;\lambda,p). \tag{13}$$

For this study, we also need the following special numbers and polynomials:
The Frobenius-Euler polynomials $H_v(s;u)$ are defined by

$$\frac{e^{su}}{e^u-\eta} = \sum_{v=0}^{\infty}\frac{H_v(s;\eta)}{(1-\eta)\,v!}u^v, \tag{14}$$

which yields

$$H_v(\eta) := H_v(0;\eta)$$

denotes the Frobenius-Euler numbers (or Eulerian numbers, or Euler-Frobenius numbers), and also

$$H_v(-1) := E_v,$$

which denotes the Euler numbers (*cf.* [33, 41]).

The Apostol-Bernoulli polynomials $\mathcal{B}_v(s;\eta)$ are defined by

$$\frac{ue^{su}}{\eta e^u-1} = \sum_{v=0}^{\infty}\frac{\mathcal{B}_v(s;\eta)}{v!}u^v, \tag{15}$$

which yields

$$\mathcal{B}_v(\eta) := \mathcal{B}_v(0;\eta)$$

denotes the Apostol-Bernoulli numbers and for $\eta=1$,

$$\mathcal{B}_v(s,1) := B_v(s)$$

denotes the Bernoulli polynomials (*cf.* [1, 33, 41]).

Let $j \in \mathbb{N}_0$. The Stirling numbers of the second kind $S(c,j)$ are defined by

$$(e^w-1)^j = \sum_{c=0}^{\infty}S(c,j)\frac{j!w^c}{c!}, \tag{16}$$

(*cf.* [1]-[41]).

We now give some survey on the Eulerian numbers, which are firstly given by combinatorial definition with the notations of the book of Petersen [23]:

For a given positive integer n, the symmetric group S_n is the set of all permutations of $[n] = \{1, 2, \ldots, n\}$, i.e., bijections $w : [n] \to [n]$. One can write permutations in one-line notation: $w = w(1)w(2)\cdots w(n)$, so a typical element of S_7 is $w = 3125647$. For any permutation $w \in S_n$, we define a descent to be a position j such that $w(j) > w(j+1)$, and we denote by $\text{Des}(w)$ the set of descents of w,

$$\text{Des}(w) = \{j : w(j) > w(j+1)\}.$$

Let $\text{des}(w)$ denote the number of descents of w, which is given by

$$\text{des}(w) = |\text{Des}(w)| = \#\,\text{Des}(w),$$

where

$$|\text{Des}(w)| = |\{j : w(j) > w(j+1)\}|\,.$$

In order to understand the above definition, we now give the following example:

If $w = 23125647$, then there are descents in position 1 since $3 > 1$, and in position 5 since $6 > 4$. Thus, there are 2 descents in position. Therefore,

$$\text{des}(w) = \text{des}(13125647) = 2.$$

By the same calculations, using permutations in S_4 grouped, we have
$\text{des}(1234) = 0,\ \text{des}(3412) = \cdots = \text{des}(1243) = 1,\ \text{des}(4213) = \cdots = \text{des}(2431) = 2,\ \text{des}(4321) = 3.$

The permutation $12\ldots n$ is the only permutation with no descents, while its reversal, $n\ldots 21$, has the maximal number, with $n-1$. It is time to give definition the Eulerian numbers, which are denoted by $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle$, with the aid of the definition of $\text{des}(w)$. The Eulerian numbers $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle$ is the number of permutations in S_n with j descents, which is given by

$$\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle = |\{w \in S_n : \text{des}(w) = j\}|$$

(*cf.* [23]).

For $n = 4$, there are 11 permutations in S_4 with two descents. Hence, $\left\langle \begin{matrix} 4 \\ 2 \end{matrix} \right\rangle = 11.$

The Eulerian polynomials are defined by the definition des as follows:

$$\mathcal{A}_n(\lambda) = \sum_{w \in S_n} \lambda^{\text{des}(w)} = \sum_{v=0}^{n-1} \left\langle \begin{matrix} n \\ v \end{matrix} \right\rangle \lambda^v, \tag{17}$$

where $n > 0$ and $\mathcal{A}_0(\lambda) = 1$ (*cf.* [23]).

The Eulerian polynomials are also defined by means of the following generating function:

$$\frac{1-\lambda}{1-\lambda e^{w(1-\lambda)}} = \sum_{d=0}^{\infty} \mathcal{A}_d(\lambda) \frac{w^d}{d!},$$

where

$$\mathcal{A}_d(\lambda) = \sum_{e=1}^{d} \left\langle \begin{matrix} d \\ e \end{matrix} \right\rangle \lambda^e$$

and

$$\left\langle \begin{matrix} d \\ e \end{matrix} \right\rangle = \sum_{c=0}^{e} (-1)^c \binom{d+1}{c} (1+e-c)^d$$

which are given the number of permutations of $\{1, 2, \ldots, d\}$ with permutation $e-1$ ascents (*cf.* [2, 6, 7, 15], [27, p. 391, Lemma 7.1]).

The number of ascents of a permutation w, $\mathrm{asc}(w) = \{j : w(j) < w(j+1)\}$, e.g., $\mathrm{asc}(1374265) = 3$.

By using the above definitions, we have $\mathcal{A}_0(\lambda) = 0$, $\mathcal{A}_1(\lambda) = 1$, $\mathcal{A}_2(\lambda) = 1+\lambda$, $\mathcal{A}_3(\lambda) = 1 + 4\lambda + \lambda^2$, and $\mathcal{A}_4(\lambda) = 1 + 11\lambda + 11\lambda^2 + \lambda^3$ and so on.

The Narayana number $N_{n,k}$ to be the number of permutations in $S_n(231)$ with k descents:

$$N_{n,k} = \{w \in S_n(231) : \mathrm{des}(w) = k\} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$$

and

$$\mathcal{N}_n(\lambda) = \sum_{w \in S_n(231)} \lambda^{\mathrm{des}(w)} = \sum_{v=0}^{n-1} N_{n,v}\lambda^v$$

(*cf.* [23]).

The Catalan numbers

$$C_n = |S_n(231)|,$$

where $n > 0$ (*cf.* [23]).

Formulas for the Euler-Frobenius polynomials and exponential Euler spline

In [37], by joining (6) with (14), for $p = 1$, $\beta = -1$ and $\lambda = -\mu$, we gave

$$B_v(-1; -\mu, 1) = \sum_{m=0}^{\infty} m^v \mu^m, \tag{18}$$

where $|\mu| < 1$. It is well-known that (18) is related to the Eulerian numbers, the B-spline and exponential Euler spline. That is, for $|\lambda| < 1$,

$$\sum_{m=0}^{\infty} m^v \lambda^m = \frac{\mathcal{A}_n(\lambda)}{(1-\lambda)^{v+1}}, \tag{19}$$

$$\sum_{m=0}^{\infty} (m+1)^v \lambda^m = \frac{\Pi_v(\lambda)}{(1-\lambda)^{v+1}},$$

where

$$\frac{1}{\lambda - e^u} = \sum_{v=0}^{\infty} \frac{\Pi_v(\lambda)}{(\lambda-1)^{v+1}} \frac{u^v}{v!} \tag{20}$$

(*cf.* [27, p. 391, Lemma 7.1]).

Combining (20) with (15), we get

$$-\frac{1}{\lambda}\sum_{v=0}^{\infty} \frac{\mathcal{B}_v(0;\lambda)}{v!} u^v = \sum_{v=0}^{\infty} \frac{\Pi_v(\lambda)}{v!\,(\lambda-1)^{v+1}} u^{v+1}$$

(*cf.* [37]). Therefore

$$\Pi_{v-1}(\lambda) = -\frac{(\lambda-1)^v}{\lambda}\,\mathcal{B}_v(0;\lambda),$$

where $v \in \mathbb{N}$ (*cf.* [37]).

Therefore, we showed that special values of $B_v(\beta; \lambda, p)$ are generalization of the interpolation functions for the Euler-Frobenius polynomials $\Pi_v(\lambda)$ and the Eulerian polynomials (*cf.* [37]).

Applying m-times derivative operator to geometric series

In this section, by applying m-times derivative operator to geometric series, we give some finite sums including the Eulerian polynomials and the sum $B_v(\beta;\lambda,p)$.

Differentiating the following geometric series m-times with respect to μ:

$$\sum_{k=0}^{\infty}\mu^k=\frac{1}{1-\mu},$$

where $|\mu|<1$, we have

$$\frac{m!}{(1-\mu)^{m+1}}=\sum_{k=0}^{\infty}k(k-1)(k-2)\cdots(k-m+1)\mu^{k-m}.$$

From the above equation, we also arrive at the binomial theorem:

$$\frac{t^v}{(1-\mu)^{m+1}}=\sum_{k=0}^{\infty}\binom{k+m-v}{m}\mu^k \tag{21}$$

(*cf.* [23]). By using the above binomial series Petersen [23] defined

$$s_k(\mu)=(1-\mu)^{k+1}\sum_{v=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}v^j\mu^v. \tag{22}$$

Differentiating the equation (22), we have

$$s_{k+1}(\mu)=(1+k\mu)s_k(\mu)+\mu(1-\mu)\frac{d}{d\mu}\left\{s_k(\mu)\right\}.$$

Using the above equation, for $\mathcal{A}_k(\lambda)=s_k(\mu)$, with the aid of (18) and (19), we have the following Carlitz identity:

$$\sum_{v=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}v^j\mu^v=\frac{\mathcal{A}_k(\lambda)}{(1-\mu)^{k+1}}, \tag{23}$$

where k is a nonnegative integer (*cf.* [23, p. 10]).

Combining the above equation with (17), we have

$$\sum_{v=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}v^j\mu^v=\sum_{v=0}^{k-1}\left\langle\begin{matrix}k\\v\end{matrix}\right\rangle\frac{\lambda^v}{(1-\mu)^{k+1}}.$$

Combining the above equation with (21), we have

$$\sum_{m=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}k^j\mu^m=\sum_{m=0}^{\infty}\sum_{v=0}^{k-1}\left\langle\begin{matrix}k\\v\end{matrix}\right\rangle\binom{k+m-v}{k}\mu^m.$$

By equalizing the coefficients of μ^m on both sides of the previous equation, we arrive at the following well-known Worpitzky's identity:

$$\sum_{j=0}^{k}\binom{k}{j}k^j=\sum_{v=0}^{k-1}\left\langle\begin{matrix}k\\v\end{matrix}\right\rangle\binom{k+m-v}{k}.$$

Combining (18) with (23), we get the following theorem:

Theorem 1. *Let v be a positive integer. Then we have*

$$B_v(-1; -\mu, 1) = \frac{1}{(1-\mu)^{v+1}} \sum_{w \in S_v} \mu^{\mathrm{des}(w)}, \tag{24}$$

where $|\mu| < 1$.

Combining (24) with (17), we also (19).

Conclusion

Here we note that des is a function. This function is given by

$$\mathrm{des} : \cup_{n \geq 1} S_n \to \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

The function des is an example of a permutation statistic, which is a function from the set of permutations to the integers. The permutation statistics include both the number of inversions and the number of cycles. When one counts permutations according to a particular permutation statistic, this gives rise to the distribution of the statistic. Any statistic whose distribution gives the numbers $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle$ is called an Eulerian statistic. Therefore, we arrive at the following open questions:

How one can find relations between the Eulerian statistic and the infinite sum $B_v(\beta; \lambda, p)$, with special values of β, λ, and p?

How one can find relations among the Narayana polynomials $\mathcal{N}_n(\lambda)$, the Catalan numbers, the Eulerian spline and the infinite sum $B_v(\beta; \lambda, p)$, with special values of β, λ, and p?

Is there any effect of the Eulerian spline in the Eulerian statistic?

Our future plan is not only to investigate the above open questions, but also other relations among the certain sums involving higher powers of (inverse) binomial coefficients with their generating functions and the family of zeta functions and also exponential Euler type spline.

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Boundary handling for biorthogonal wavelets

Ahmet Alturk *[1] *and Fritz Keinert* [2]

Wavelets are new families of basis functions that can be used to decompose and reconstruct functions defined on $\mathbb{R}$. Standard wavelet theory mostly considers functions on the whole real line. In practice, however, we work with functions defined on compact intervals. Several different approaches exist to overcome this problem in the literature. One of the important and useful approach involves using boundary functions. In this work, we use boundary functions that are defined by recursion relations and obtain regularity properties directly from the recursion relation for biorthogonal wavelets.

2020 MSC: 45B05, 33C45, 47A52

KEYWORDS: Wavelets, Orthogonal wavelets, Biorthogonal wavelets

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Fractional periodic discontinuous Sturm-Liouville problem with eigenparameter in the boundary condition

Abdullah Kablan

The development of the theory of fractional calculus has opened further perspectives for the theory of new-type problems, called fractional Sturm-Liouville problems. This study investigates the some fundemantal theorems for fractional periodic discontinuous Sturm-Liouville problem with eigenparameter in the boundary condition. The definition of a new inner product space was one of the most significant cornerstones of this work. We were able to prove the theorems in this manner.

2020 MSC: 34B24, 26A33, 34B08

KEYWORDS: Sturm-Liouville problems, Fractional derivative, Transmission conditions

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Further properties of a^*-$\boldsymbol{I}$-open sets and a^*-$\boldsymbol{I}$-continuity

Aynur Keskin Kaymakci

In this paper, first of all we investigate some furher properties of a^*-I-open set. The most important of them, the family of all a^*-I-open sets in any ideal topological space forms a supra topology. Then, we give a characterization of a^*-I-closed set which is complement of a^*-I-open. Finally, we obtain essential properties of a^*-I-continuity and the two properties are called *aI-compactness* and *aI-connectedness* that are not preserved under this function, even if the function is surjective for first one.

2020 MSC: 54 A05, 54 B05, 54 C05

KEYWORDS: a^*-I-open set, a^*-I-continuity, Ideal

Introduction

Ideals or set ideals in topological spaces has been studied by Kuratowski [6] and Vaidyanathaswamy [12]. Jankovic and Hamlett [4] studied related this subject and defined a new topological space called ideal topological space using ideal. On the other hand, continuity and generalized continuity are important in several areas of mathematics and related sciences as well as topology. In this paper, we investigate depply some properties of an a^*-I-set and a^*-I-continuous function.

Preliminaries

Throughout this paper, we will denote any topological spaces by (X, τ), (Y, φ) and (Z, ψ). For a subset A of a space (X, τ), the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. It is well known that a subset A of a space (X, τ) is said to be *regular open* (resp. *regular closed*) [11] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A subset A of a space (X, τ) is said to be δ-*open* [13] if for each $x \in A$ there exists a regular open set U such that $x \in U \subseteq A$. A is δ-*closed* [13] if (X-A) is δ-*open*. The set $\{x \in X \mid x \in U \subseteq A$ for some regular open set U of $X\}$ is called the δ-interior of A and is denoted by $\delta Int(A)$ [13]. A point $x \in X$ is called a δ-cluster point of A if $A \cap Int(Cl(V)) \neq \varnothing$ for each open set V containing x [13]. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta Cl(A)$ [13]. Of course, δ-open sets form a topology τ^δ and then $\tau^\delta \subset \tau$ holds.

An ideal [6], [12] I on X is defined as a nonempty collection of subsets of X which ensures: (1) If $A \in I$ and $B \subset A$, then $B \in I$, (2) If $A \in I$ and $B \in I$, then $(A \cup B) \in I$.

Applications to various fields were futher investigated by several authors such that Mukherje et al. [8]. Let (X, τ) be a topological space and I an ideal on X. An ideal topological space [4] is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I, \tau) = \{x \in X \mid U \cap A \notin I$ for each open neighbourhood U of $x\}$ is called the local function of A [6] with respect to I and τ. Throught this paper, we use A^* instead of $A^*(I, \tau)$. Besides, in [4], authors introduced a new Kuratowski closure operator $Cl^*(.)$ defined by $Cl^*(A) = A \cup A^*$ and obtained a new topology on X which is called an $*$-topology. This topology is denoted by $\tau^*(I)$ briefly, τ^* which is finer than τ i.e., $\tau \subset \tau^*$.

A point x in an ideal topological space (X, τ, I) is called *δ_I-cluster point* of A if $Int(Cl^*(U) \cap A \neq \varnothing$ for each neighborhood U of x. The set of all *δ_I-cluster points* of A is called the *δ_I-closure of* A and will be denoted by $\delta Cl_I(A)$ [15]. A is said to be *δ_I-closed* [15] if $A = \delta Cl_I(A)$. Of course, the complement of δ_I-closed set is said *δ_I-open* [15]. The family of all *δ_I-open* sets in any ideal topological space (X, τ, I) form a topology and denoted by $\tau^{\delta I}$. Besides, $\tau^{\delta} \subset \tau^{\delta I}$ and $\tau^{\delta I} \subset \tau$ hold. It is known that a subset A in an ideal topological space (X, τ, I) is said to be *R-I-open* (resp. *R-I-closed*) [15] if $A = Int(Cl^*(A))$ (resp. $A = Cl^*(Int(A))$). The set *δ_I-interior* of A is the union of all *R-I*-open sets of X contained in A and is denoted by $\delta Int_I(A)$. Now, we remember necessary some definitions and properties for this paper.

Lemma 1. *Let (X, τ, I) be an ideal topological space and A, B, U subsets of X. Then the following properties are hold:*
(1) If $A \subset B$, then $A^* \subset B^*$ ([4]),
(2) If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$ ([4]),
(3) If $U \in \tau$, then $U \cap Cl^*(A) \subseteq Cl^*(U \cap A)$ ([2]).

Definition 2 (*cf.* [1])**.** *A subset of A in an ideal topological space (X, τ, I) is a topological space with an ideal $I_A = \{S \cap A : S \in I\}$. Then, I_A is an ideal on A.*

Definition 3 (*cf.* [4])**.** *Let (X, τ, I) be an ideal topological space and A, B subsets of X such that $B \subset A$. Then*

$$B^*(\tau_A, I_A) = B^*(\tau, I) \cap A.$$

Lemma 4 (*cf.* [9])**.** *Let $f : (X, \tau) \to (Y, \varphi)$ be a function and I be an ideal on X, then f(I) is an ideal on Y and it is defined by f(I) $= \{f(S) : S \in I\}$.*

Definition 5 (*cf.* [3])**.** *A subset A of an ideal topological space (X, τ, I) is said to be $\delta_I -$ dense if $\delta Cl_I(A) = X$.*

Definition 6 (*cf.* [5])**.** *A subset A of an ideal topological space (X, τ, I) is said to be an a^*-I-open if $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$.*

The family of all a^*-I-open subsets of an ideal topological spaces (X, τ, I) is denoted by $a * IO(X, \tau)$ ([5]).

Lemma 7 (*cf.* [5])**.** *$\varnothing$ and X are both a^*-I-open sets.*

a^*-I-open sets

In this section, we obtain some properties of a^*-I-open set such as the family all of a^*-I-open sets in any ideal topological space forms a supratopology. Besides, we give a characterization of a^*-I-closed which is complement of a^*-I-open. Finally we introduce new interior and closure operators using these sets, respectively.

Proposition 8 (*cf.* [5])**.** *Let (X, τ, I) be an ideal topological space with an arbitrary index set Δ. If $\{A_\alpha : \alpha \in \Delta\} \subset a^*IO(X, \tau)$, then $\cup\{A_\alpha : \alpha \in \Delta\} \in a^*IO(X, \tau)$.*

Remark 2. *The intersection of two a^*-I-open sets need not be an a^*-I-open set as shown by the next example.*

Example 9. *Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\varnothing, \{a\}\}$. Then $A = \{a, d\}$ and $B = \{c, d\}$ are a^*-I-open sets, but their intersection $A \cap B = D = \{d\}$ is not a^*-I-open.*

Remark 3. *The family of a^*-I-open sets doesn't form a topology.*

Definition 10. *The family of γ all subsets X is called supra topology ([7]) if the finite intersection property is excluded from the topology.*

Proposition 11. *For an ideal topological space (X, τ, I), the family of all a^*-I-open sets forms a supra topology.*

Proof. This proof is obtained from Lemma 1.7, Proposition 2.1, Remark 1 and Definition 2.3. □

Lemma 12 (*cf.* [5])**.** *Let (X, τ, I) be an ideal topological space and A, U are subsets of X. If A is an a^*-I-open set and U is δ-I-open set. Then $(A \cap U)$ is an a^*-I-open.*

Since $\tau^{\delta I} \subset \tau$, we obtain one result better than above lemma.

Proposition 13. *Let (X, τ, I) be an ideal topological space and A, U be two subsets of X. If A is an a^*-I-open set and U is open set, then $(A \cap U)$ is an a^*-I-open set.*

Proof. Since A is an a^*-I-open set and U is an open set, we have $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$ and $U \subset \delta Int(U)$. By using some properties of closure, interior and δ-I-closure operations, we have

$$\begin{aligned}(A \cap U) &\subset [Int(\delta Cl_I(A)) \cup Cl(Int(A))] \cap Int(U) \\ &= [Int(\delta Cl_I(A)) \cap Int(U)] \cup [Cl(Int(A)) \cap Int(U)] \\ &\subseteq Int[\delta Cl_I(A) \cap Int(U)] \cup Cl[Int(A) \cap Int(U)] \\ &\subseteq Int[\delta Cl_I(A \cap Int(U)] \cup Cl[Int(A) \cap Int(U)] \\ &\subseteq Int[\delta Cl_I(A \cap Int(U))] \cup Cl(Int(A \cap U)) \\ &\subseteq Int(\delta Cl_I(A \cap U)) \cup Cl(Int(A \cap U)).\end{aligned}$$

This shows that $(A \cap U)$ is an a^*-I-open set. □

Proposition 14. *Let (X, τ, I) be an ideal topological space and A, U are subsets of X. If A is an a^*-I-open set and U is open set. Then $(A \cap U)$ is an a^*-I-open set in (U, τ_U, I_U).*

Proof. This proof is obtained from Proposition 2.6. □

One can obtain directly the next result using Definitions 1.5 and 1.6.

Corollary 15. *Every $\delta_I -$ dense is an a^*-I-open.*

Definition 16 (*cf.* [5])**.** *A subset A of an ideal topological space (X, τ, I) is said to be an a^*-I-closed if $(X - A)$ is an a^*-I-open.*

Theorem 17. *A subset A of an ideal topological space (X, τ, I) is an a^*-I-closed, then $Cl(\delta Int_I(A)) \cap Int(Cl(A)) \subset A$.*

Proof. Since A is an a^*-I-closed, then $(X - A)$ is an a^*-I-open. By using $\tau^{\delta} \subset \tau^{\delta I} \subset \tau \subset \tau^*$, we have

$$\begin{aligned}(X - A) &\subset [Int(\delta Cl_I(X - A)) \cup Cl(Int(X - A))] \\ &\subseteq [Int(\delta Cl(X - A)) \cup Cl(Int(X - A))] \\ &= [X - Cl(\delta Int(A))] \cup [X - (Int(Cl(A)))] \\ &\subseteq [X - Cl(\delta Int_I(A))] \cup [X - (Int(Cl(A)))] \\ &\subseteq (X - [Cl(\delta Int_I(A)) \cap Int(Cl(A))]).\end{aligned}$$

Therefore, $(X - A) \subset (X - [Cl(\delta Int_I(A)) \cap Int(Cl(A))])$ and hence $[Cl(\delta Int_I(A)) \cap Int(Cl(A))] \subset A$. □

Corollary 18. *A subset A of an ideal topological space (X, τ, I) such that $[X - Cl(\delta Int_I(A))] = [Int(\delta Cl_I(X - A))]$. Then A is an a^*-I-closed if and only if $Cl(\delta Int_I(A)) \cap Int(Cl(A)) \subset A$.*

Proof. Since both necessity condition is given, we only prove the sufficient condition. Let $Cl(\delta Int_I(A)) \cap Int(Cl(A)) \subset A$. Then

$$\begin{aligned}(X - A) &\subset (X - [Cl(\delta Int_I(A)) \cap Int(Cl(A))]) \\ &= [X - Cl(\delta Int_I(A))] \cup [X - (Int(Cl(A)))] \\ &\subseteq (Int(\delta Cl_I(X - A))) \cup (Cl(Int(X - A))).\end{aligned}$$

This shows that $(X - A)$ is an a^*-I-open and hence A is an a^*-I-closed. □

Proposition 19. *Let A, B be two subsets of ideal topological space (X, τ, I) such that A is an a^*-I-open and B is an a^*-I-closed in (X, τ, I). Then there exist a^*-I-open U and a^*-I-closed V such that $(A \cap B) \subset V$ and $U \subset (A \cup B)$.*

Proof. Let $V = (Cl_a^*(A) \cap B)$ and $U = (A \cup Int_a^*(B))$. Then, V is an a^*-I-closed and U is an a^*-I-open. So, $A \subset Cl_a^*(A)$ implies $(A \cap B) \subset (Cl_a^*(A) \cap B) = V$ and $Int_a^*(B) \subset B$ implies $(A \cup Int_a^*(B)) = U \subset (A \cup B)$. □

We introduce two operations are related to a^*-I-open and a^*-I-closed, respectively. Obvious that they are new supra-interior and supra-closure operators.

Definition 20. *Let A be a subset of ideal topological space (X, τ, I).*

(1) The union of all a^*-I-open contained in A is called a-I-interior of A and its denoted by $Int_a^*(A)$,

(2) The intersection of all a^*-I-closed containing A is called a-I-closure of A and its denoted by $Cl_a^*(A)$.

Remark 4. *Let A, B be two subsets of ideal topological space (X, τ, I). Then, the following equivalents are hold:*

(1) A is an a^*-I-open if and only if $Int_a^*(A) = A$,

(2) B is an a^*-I-closed if and only if $Cl_a^*(B) = B$.

Properties of a^*-I-continuous functions

In this section, we obtain some properties of a^*-I-continuity.

Definition 21 (*cf.* [5])**.** *A function $f : (X, \tau, I) \to (Y, \varphi)$ is said to be an a^*-I-continuous if $f^{-1}(V)$ is an a^*-I-open set for every open set V in (Y, φ).*

Proposition 22. *Let $f : (X, \tau, I) \to (Y, \varphi)$ and $g : (Y, \varphi) \to (Z, \psi)$ be two functions. Then gof is an a^*-I-continuous if f is an a^*-I-continuous and g is continuous.*

Proof. This proof is obtained from necessary definition. □

Proposition 23. *Let $f : (X, \tau, I) \to (Y, \varphi)$ be is an a^*-I-continuous and $U \in \tau$. Then the restriction $f_U : (U, \tau_U, I_U) \to (Y, \varphi)$ is an a^*-I-continuous.*

Proof. Let V be any open set of (Y, φ). Since f is an a^*-I-continuous, $f^{-1}(V) \in a * IO(X, \tau)$. We have $f_U^{-1}(V) = (f^{-1}(V) \cap U) \in a * IO(U, \tau_U)$ by using Definition 1.2, Definition 1.3 and Proposition 2.7. Finally, we have $f_U : (U, \tau_U, I_U) \to (Y, \varphi)$ is an a^*-I-continuous. □

We have the next theorem without proof.

Theorem 24. *Let $f : (X, \tau, I) \to (Y, \varphi)$ be a function and $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. If the restriction function f_{U_α} is an a^*-I-continuous for each $\alpha \in \Delta$, then f is an a^*-I-continuous.*

Definition 25 (*cf.* [9, 10])**.** *A subset A of an ideal topological space (X, τ, I) is said to be I-compact if for every τ-open cover $\{W_\alpha : \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $(X - \cup\{W_\alpha : \alpha \in \Delta_0\}) \in I$.*

Definition 26. *An ideal topological space (X, τ, I) is called aI-compact if for every a^*-I-open cover $\{W_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $(X - \cup\{W_\alpha : \alpha \in \Delta_0\}) \in I$.*

Theorem 27. *The image of aI-compact space under a^*-I-continuous surjective function is f(I)-compact.*

Proof. Let $f : (X, \tau, I) \to (Y, \varphi)$ be an a^*-I-continuous surjective function and $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Then, we have $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is an a^*I-open cover of X. Since (X, τ, I) is aI-compact, then there exists finite subset of Δ_0 of Δ such that $(X - \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}) \in I$. Hence, we have $(Y - \cup\{(V_\alpha) : \alpha \in \Delta_0\}) \in f(I)$. This shows that $(Y, \varphi, f(I)$ is *f(I)-compact*. □

Definition 28. *A connected space* [14] *is a topological space that cannot be represented as the union of two disjoint non-empty open subsets.*

Definition 29. *An ideal topological space (X, τ, I) is said to be aI-connected if it cannot be represented as the union of two disjoint non-empty a^*I-open subsets.*

Theorem 30. *The image of aI-connnected space under a^*-I-continuous function is connected.*

Proof. Let $f : (X, \tau, I) \to (Y, \varphi)$ be an a^*-I-continuous function and (X, τ, I) be an aI-connnected space. Assume that (Y, φ) be disconnected. Then U and V are open and $Y = U \cup V$, where $U \cap V = \emptyset$. By hypothesis since f is a^*-I-continuous, $X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty a^*-I-open sets in (X, τ, I). Besides, $(f^{-1}(U) \cap f^{-1}(V)) = \emptyset$. Therefore we have (X, τ, I) isn't aI-connected, which is contradiction. So, (Y, φ) is connected. □

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Leukemia disease model

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The main objective of this paper is to analyze a hybrid mathematical model describing the dynamics of hematopoietic stem cell population in the chronic myeloid leukemia. First, we formulate the basic model as a one which contain two equations, one of them represent an age structured equation and the other one is an ordinary differential equation. Next to understand the behavior of our solutions we study the existence of steady state and their stability. Finally in order to confirm our results we give some simulations.

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Introduction

The chronic myeloid leukemia (CML) is also known as chronic myelogenous leukemia. It's a type of cancer that starts in certain blood-forming cells of the bone marrow. In 1960 the molecular responsible of CML was known as the philadelphia chromosome (Ph), the scientists discovered this molecular as the chromosome 22 and they describe it as a short one, the translocation of chromosome 9 and 22 happened in 1973 (see [6, 7, 14, 15]).
To understand the dynamics of chronic myeloid Leukemia, many studies have been conducted (see [2, 3, 5, 7]).
Among these models, the authors in [3] have studied a model for the growth of hematopoietic cells represented by a system of ordinary differential equations

$$\begin{cases} \dfrac{dx_0}{dt} = n\phi\left[\epsilon_1(x_0+y_0)+\epsilon_2(x_1+y_1)\right]x_0 - d_0x_0, & t\in(0,T) \\ \dfrac{dx_1}{dt} = rx_0 - d_1x_1, & t\in(0,T) \\ \dfrac{dy_0}{dt} = m\psi\left[\epsilon_1(x_0+\alpha y_0)+\epsilon_2(x_1+\alpha y_1)\right]y_0 - g_0y_0, & t\in(0,T) \\ \dfrac{dy_1}{dt} = qy_0 - g_1y_1, & t\in(0,T). \end{cases} \tag{1}$$

In the model (1), $x_0(t)$, $x_1(t)$, $y_0(t)$ and $y_1(t)$ denotes respectively normal stem cells, differentiated normal stem cells, leukemia stem cells and differentiated leukemic stem cells at time $t \in (0,T)$. The mortality rate of the cells is given by d_0, d_1, g_0 and g_1 for each compartment. The division rate of $x_0(t)$ and $y_0(t)$ is n, m respectively. Normal stem cells and leukemia stem cells produce differentiated stem cells with a rate r and q respectively. The homeostasis of normal stem cells and leukemia stem cells is achieved by the functions ϕ and ψ.
The mathematical analysis of the model (1) includes three scenarios. In the first

scenario, $\epsilon_1 = 1$ and $\epsilon_2 = 0$. In the second one, $\epsilon_1 = 0$ and $\epsilon_2 = 1$. In the last one, $\epsilon_1 = \epsilon_2 = 1$.
An age structured model has been studied in [2, 5]

$$\begin{cases} \dfrac{\partial x}{\partial t} + \dfrac{\partial x}{\partial a} = -d_0(a)x, & (t,a) \in (0,T) \times (0,A), \\ \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} = -g_0(a)y, & (t,a) \in (0,T) \times (0,A), \\ x(t,0) = \displaystyle\int_0^A \phi\left(\int_0^A k_1(a,\hat{a})(x+y)(t,\hat{a})\,\mathrm{d}\hat{a}\right) x(t,a)\,\mathrm{d}a, & t \in [0,T], \\ y(t,0) = \displaystyle\int_0^A \psi\left(\int_0^A k_2(a,\hat{a})(x+\alpha y)(t,\hat{a})\,\mathrm{d}\hat{a}\right) y(t,a)\,\mathrm{d}a, & t \in [0,T], \\ x(0,a) = \varphi_1(a), \\ y(0,a) = \varphi_2(a), \end{cases} \tag{2}$$

where $x(t,a)$, $y(t,a)$ denotes respectively size of normal stem cells and leukemia stem cells at time $t \in (0,T)$ and age $a \in (0,A)$. The death rate is given by $d_0(a)$ and $g_0(a)$ for each compartment. The sentence of homeostasis is achieved by the functions ϕ and ψ with k_1, k_2 denotes coefficients of interaction.
The authors in [2] have studied the model (2) through the calculate of optimal control. On the other hand the authors in [5] have studied the behavior of the solution through the calculate of steady states.
Inspired by the models (1) and (2) in the works [2, 3, 5], we are interested in the study of an hybrid mathematical model of chronic myeloid leukemia using the following system

$$\begin{cases} \dfrac{\partial u_1(t,a)}{\partial t} + \dfrac{\partial u_1(t,a)}{\partial a} = -\mu_1(a)u_1(t,a), & (t,a) \in (0,T) \times (0,A), \\ \dfrac{du_2}{dt} = \left[m\psi(U_1(t) + \alpha u_2(t)) - g_0\right] u_2(t), & t \in [0,T], \\ u_1(t,0) = \displaystyle\int_0^A \tilde{\varphi}_1\left(a, k_1\left[\int_0^A u_1(t,a)\,\mathrm{d}a + u_2(t)\right]\right) u_1(t,a)\,\mathrm{d}a, & t \in [0,T], \\ u_1(0,a) = \phi_1(a), & a \in [0,A], \\ u_2(0) = \phi_2, \end{cases} \tag{3}$$

where
$u_1(t,a)$: size of normal stem cells at time $t \in (0,T)$ and age $a \in (0,A)$,
$u_2(t)$: size of leukemia stem cells at time $t \in (0,T)$,
$U_1(t) = \displaystyle\int_0^A u_1(t,a)\,\mathrm{d}a$, denote the total size of normal stem cells,
$\phi_1(a)$: initial condition of normal stem cells,
ϕ_2: initial condition of leukemia stem cells,
$\mu_1(a)$: death rate of normal stem cells,
g_0: death rate of leukemia stem cells,
m: rate division of leukemia stem cells.
Depending on Hill functional, the homeostasis of normal and leukemic stem cells are achieved by the following equations (see [1, 7, 12])

$$\tilde{\varphi}_1\left(a, k_1\left[\int_0^A u_1(t,a)\,\mathrm{d}a + u_2(t)\right]\right) = \frac{\theta^n \varphi_1(a)}{\theta^n + \left(k_1\left[\displaystyle\int_0^A u_1(t,a)\,\mathrm{d}a + u_2(t)\right]\right)^n}.$$

$$\psi(U_1(t) + \alpha u_2(t)) = \frac{1}{1 + c\left(U_1(t) + \alpha u_2(t)\right)^n},$$

where
$\varphi_1(a)$: rate division for normal stem cells,
k_1: coefficient of interaction,
α: competition parameter between normal and leukemic stem cells with values in $]0,1[$ (*cf.* [2, 3, 9, 10]),
θ: the crowding effect (*cf.* [12, 13]),
c: dimensionless parameter.
In [2, 4, 11, 16, 17], the authors have already proved the existence of a global solution by specifying the conditions of the parameters in (3).
The paper is organized as follows; First we study the model (3) through the calculations of the steady states. Second we study the stability of steady states. third we give some simulations and some conclusions.

Existence of steady states

In this section, we analyze the existence of the steady states of (3), which will be given by $u(a) = (u_1(a), u_2^*)$.
From the second equation of (3), we have

$$\frac{du_2^*}{dt} = 0 \ \ if \ \ and \ \ only \ \ if \ \ \left[\frac{m}{1 + c\left(\int_0^A u_1(a)\,\mathrm{d}a + \alpha u_2^*\right)^n} - g_0 \right] u_2^* = 0. \tag{4}$$

This implies

$$u_2^* = 0$$

or

$$u_2^* = \frac{1}{\alpha}\left(\sqrt[n]{\frac{m - g_0}{c \ g_0}} - \int_0^A u_1(a)\,\mathrm{d}a \right) = \frac{1}{\alpha}\left(\sqrt[n]{\frac{m - g_0}{c \ g_0}} - U_1^* \right),$$

where $U_1^* = \int_0^A u_1(a)\,\mathrm{d}a$, with $0 \leq U_1^* < \sqrt[n]{\frac{m - g_0}{c \ g_0}}$.

From the first equation of (3), we have

$$\frac{du_1}{da} = -\mu_1(a) u_1(a) \tag{5}$$

with

$$u_1(0) = \int_0^A \left(\frac{\theta^n \varphi_1(a) u_1(a)}{\theta^n + \left(k_1 \left[\int_0^A u_1(a)\,\mathrm{d}a + u_2^*\right]\right)^n} \right) \mathrm{d}a \tag{6}$$

From (5) we have $u_1(a) = u_1(0)\pi_1(a)$, with $\pi_1(a) = e^{-\int_0^a \mu_1(s)ds}$ is the survival probability.
If $u_1(0) = 0$ then $u_1(a) = 0$, so we have two steady states.
The trivial steady state $u^0 = (0, 0)$ and the blast steady state

$$u^b = \left(0, \frac{1}{\alpha}\left(\sqrt[n]{\frac{m - g_0}{c \ g_0}} \right)\right)$$

.
If $u_1(0) \neq 0$ and $u_2^* = 0$, then from (6), we have

$$1 = \int_0^A \left(\frac{\theta^n \varphi_1(a)\pi_1(a)}{\theta^n + \left(k_1 u_1(0) \int_0^A (\pi_1(a)\,\mathrm{d}a \right)^n} \right) \mathrm{d}a. \tag{7}$$

Put $L_1 = \int_0^A \pi_1(a)\,\mathrm{d}a$, from (7) we have

$$1 = \int_0^A \left(\frac{\theta^n \varphi_1(a)\pi_1(a)}{\theta^n + (k_1 u_1(0) L_1)^n} \right) \mathrm{d}a. \tag{8}$$

Then, from (8) we have
$\theta^n + (k_1 u_1(0) L_1)^n = \theta^n R_1$, where $R_1 = \int_0^A \varphi_1(a)\pi_1(a)\,\mathrm{d}a$ is the net reproduction rate.
We obtain

$$(k_1 u_1(0) L_1) = \theta \sqrt[n]{R_1 - 1}. \tag{9}$$

Note that a necessary condition to have a solution for (9) is that the net reproduction rate $R_1 > 1$, then we have $u_1(0) = \frac{\theta \sqrt[n]{R_1 - 1}}{k_1 L_1}$, so the third steady state is the non pathological steady state $u^{np} = \left(\frac{\theta \sqrt[n]{R_1 - 1}}{k_1 L_1} \pi_1(a), 0 \right)$.
If $u_1(0) \neq 0$ and $u_2^* \neq 0$, with $u_2^* = \frac{1}{\alpha} \left(\sqrt[n]{\frac{m - g_0}{c\ g_0}} - U_1^* \right)$
From (6), we have

$$1 = \int_0^A \left(\frac{\theta^n \varphi_1(a)\pi_1(a)}{\theta^n + [k_1(u_1(0) L_1 + u_2^*)]^n} \right) \mathrm{d}a. \tag{10}$$

From (10), we obtain

$$k_1(u_1(0) L_1 + u_2^*) = \theta \sqrt[n]{R_1 - 1}. \tag{11}$$

For $u_2^* = \frac{1}{\alpha} \left(\sqrt[n]{\frac{m - g_0}{c\ g_0}} - U_1^* \right)$, (11) becomes

$$k_1 u_1(0) L_1 + k_1 \frac{1}{\alpha} \left(\sqrt[n]{\frac{m - g_0}{c\ g_0}} - U_1^* \right) = \theta \sqrt[n]{R_1 - 1}. \tag{12}$$

Put $b_1 = \theta \sqrt[n]{R_1 - 1}$, $b_2 = \sqrt[n]{\frac{m - g_0}{c\ g_0}}$, (12) becomes

$$k_1 u_1(0) L_1 + k_1 \frac{1}{\alpha} (b_2 - u_1(0) L_1) = b_1. \tag{13}$$

From (13), we have $u_1(0) = \frac{\alpha b_1 - k_1 b_2}{k_1 L_1(\alpha - 1)}$ *then* $u_1(a) = \frac{\alpha b_1 - k_1 b_2}{k_1 L_1(\alpha - 1)} \pi_1(a)$. So the fourth steady state is the chronic steady state u^c, with $u^c = \left(\dfrac{\alpha\theta\sqrt[n]{R_1 - 1} - k_1\sqrt[n]{\dfrac{m-g_0}{c\ g_0}}}{k_1 L_1(\alpha - 1)}\pi_1(a), \dfrac{k_1\sqrt[n]{\dfrac{m-g_0}{c\ g_0}} - \theta\sqrt[n]{R_1 - 1}}{k_1(\alpha-1)} \right)$.

Now, we give the existence theorem of the steady states.

Theorem 1.

The system (3) admits the following steady states.
The trivial steady state u^0 exists always.
If $m > g_0$, then the blast steady state u^b exists.
If $R_1 > 1$, then the non pathological steady state u^{np} exists.
If $m > g_0$, and $\left(\dfrac{k_1}{\theta}\right)^n \left(\dfrac{m-g_0}{cg_0}\right) + 1 < R_1 < \left(\dfrac{k_1}{\alpha\theta}\right)^n \left(\dfrac{m-g_0}{cg_0}\right) + 1$, then the chronic steady state u^c exists.

Local stability of steady states

In this section, we study the local stability of steady states which is based on the linearization (see [5, 11, 16]) of the system (3).
Let

$$\begin{cases} x_1(t,a) = u_1(t,a) - u_1(a), \\ x_2(t) = u_2(t) - u_2^*, \end{cases} \tag{14}$$

where $(u_1(a), u_2^*)$ is one of the steady states of (3).
We derive the first equation of (14), so we obtain the following equation

$$\frac{\partial x_1}{\partial t}(t,a) + \frac{\partial x_1}{\partial a}(t,a) = -\mu_1(a)x_1(t,a). \tag{15}$$

With the boundary condition

$$x_1(t,0) = u_1(t,0) - u_1(0). \tag{16}$$

From (3), we have

$$\begin{aligned} u_1(t,0) &= \int_0^A \tilde{\varphi}_1\left(a, k_1\left[\int_0^A u_1(t,a)\,\mathrm{d}a + u_2(t)\right]\right) u_1(t,a)\,\mathrm{d}a \\ &= \int_0^A \tilde{\varphi}_1\left(a, k_1\left[\int_0^A (x_1(t,a) + u_1(a))\ \mathrm{d}a + (x_2(t) + u_2^*)\right]\right) (x_1(t,a) + u_1(a))\ \mathrm{d}a \\ &= \int_0^A \tilde{\varphi}_1\left(a, k_1\left(X_1(t) + U_1^*\right) + k_1\left(x_2(t) + u_2^*\right)\right)\left(x_1(t,a) + u_1(a)\right)\,\mathrm{d}a, \end{aligned}$$

where $X_1(t) = \int_0^A x_1(t,a)\,\mathrm{d}a$. We have

$$u_1(t,0) = \int_0^A \tilde{\varphi}_1\left[a, (k_1X_1(t) + k_1x_2(t)) + (k_1U_1^* + k_1u_2^*)\right](x_1(t,a) + u_1(a))\ \mathrm{d}a,$$

$$u_1(t,0) = \int_0^A \tilde{\varphi}_1\left(a, X(t) + \tilde{U}\right)(x_1(t,a) + u_1(a))\ \mathrm{d}a,$$

where $X(t) = k_1\left(X_1(t) + x_2(t)\right)$, $\tilde{U} = k_1\left(U_1^* + u_2^*\right)$.
We have

$$\tilde{\varphi_1}(a, X(t) + \tilde{U})(x_1(t,a) + u_1(a)) = \left[\tilde{\varphi_1}(a,\tilde{U}) + \frac{\partial\tilde{\varphi}_1}{\partial\tilde{U}}X(t) + o(X(t))\right](x_1(t,a) + u_1(a)).$$

$$= \tilde{\varphi_1}(a,\tilde{U})x_1(t,a) + \tilde{\varphi_1}(a,\tilde{U})u_1(a) + \frac{\partial\tilde{\varphi_1}}{\partial\tilde{U}}X(t)x_1(t,a)$$

$$+ \frac{\partial\tilde{\varphi_1}}{\partial\tilde{U}}X(t)u_1(a) + o(X(t)).$$

On the other hand we have $u_1(0) = \int_0^A \tilde{\varphi_1}(a,\tilde{U})u_1(a)\,\mathrm{d}a$. Then, the linearized equation of (16) at $(0,0)$ is

$$x_1(t,0) = \int_0^A \tilde{\varphi_1}(a,\tilde{U})x_1(t,a)\,\mathrm{d}a + X(t)\int_0^A \frac{\partial\tilde{\varphi_1}}{\partial\tilde{U}}u_1(a)\,\mathrm{d}a. \tag{17}$$

Now, we derive the second equation of (14)

$$\begin{aligned}\frac{dx_2}{dt} = \frac{du_2}{dt} &= \left[\frac{m}{1+c(X_1(t)+U_1^*+\alpha x_2(t)+\alpha u_2^*)^n} - g_0\right](x_2(t)+u_2^*)\\ &= \frac{mx_2(t)}{1+c(X_1(t)+U_1^*+\alpha x_2(t)+\alpha u_2^*)^n} - g_0x_2(t)\\ &+ \frac{mu_2}{1+c(X_1(t)+U_1^*+\alpha x_2(t)+\alpha u_2^*)^n} - g_0u_2^*.\end{aligned}$$

Let $\frac{dx_2}{dt} = f(X_1(t),x_2(t))$.

By a Taylor development of the function $f(X_1(t),x_2(t))$ at (0,0) we obtain the following equation

$$\begin{aligned}\frac{dx_2}{dt} &= \frac{-nmu_2^*c(U_1^*+\alpha u_2^*)^{n-1}}{[1+c(U_1^*+\alpha u_2^*)^n]^2}X_1(t)\\ &+ \left[\frac{m}{(1+c(U_1^*+\alpha u_2^*)^n)} - \frac{nmu_2^*\alpha c(U_1^*+\alpha u_2^*)^{n-1}}{(1+c(U_1^*+\alpha u_2^*)^n)^2} - g_0\right]x_2(t).\end{aligned} \tag{18}$$

Let

$$A_1 = \frac{-nmu_2^*c(U_1^*+\alpha u_2^*)^{n-1}}{[1+c(U_1^*+\alpha u_2^*)^n]^2},$$

$$A_2 = \frac{m}{(1+c(U_1^*+\alpha u_2^*)^n)} - \frac{nmu_2^*\alpha c(U_1^*+\alpha u_2^*)^{n-1}}{(1+c(U_1^*+\alpha u_2^*)^n)^2} - g_0.$$

From (15), (17) and (18) the linearized system of (3) at $u(a)$ is

$$\begin{cases}\frac{\partial x_1}{\partial t}(t,a) + \frac{\partial x_1}{\partial a}(t,a) = -\mu_1(a)x_1(t,a), & (t,a)\in(0,T)\times(0,A)\\ X(t) = k_1\left(\int_0^A x_1(t,a)\,\mathrm{d}a + x_2(t)\right) = k_1\left(X_1(t)+x_2(t)\right), & t\in[0,T]\\ x_1(t,0) = \int_0^A \tilde{\varphi_1}(a,\tilde{U})x_1(t,a)\,\mathrm{d}a + X(t)\int_0^A \frac{\partial\tilde{\varphi_1}}{\partial\tilde{U}}u_1(a)\,\mathrm{d}a, & t\in[0,T]\\ \frac{dx_2}{dt} = A_1X_1(t) + A_2x_2(t), & t\in[0,T].\end{cases} \tag{19}$$

We are looking for solutions in the form

$x_1(t,a) = f_1(a)e^{\lambda t}$, $x_2(t) = f_2e^{\lambda t}$, where $f_1(a) > 0$, $f_2 > 0$ and $\lambda\in\mathbb{C}$.

From (19), we obtain

$$\begin{cases} \dfrac{df_1}{dt} + (\lambda + \mu_1(a))\, f_1(a) = 0, \\ f_1(a) = f_1(0) e^{-\int_0^a (\lambda + \mu_1(s))\,\mathrm{d}s}, \\ F_1 = \displaystyle\int_0^A f_1(a)\,\mathrm{d}a, \\ f_1(0) = \displaystyle\int_0^A \tilde{\varphi}_1(a,\tilde{U}) f_1(a)\,\mathrm{d}a + k_1\,[F_1 + f_2] \int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a,\tilde{U}) u_1(a)\,\mathrm{d}a, \\ (\lambda - A_2) f_2 - A_1 F_1 = 0. \end{cases} \tag{20}$$

From the fourth equation of (20) we have

$$f_1(0) = \int_0^A \tilde{\varphi}_1(a,\tilde{U}) f_1(a)\,\mathrm{d}a + k_1 \left[\int_0^A f_1(a)\,\mathrm{d}a + f_2\right] \int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a,\tilde{U}) u_1(a)\,\mathrm{d}a.$$

$$\begin{aligned} f_1(0) = {} & \int_0^A \tilde{\varphi}_1(a,\tilde{U}) f_1(0) e^{-\int_0^a (\lambda+\mu_1(s))\,\mathrm{d}s}\,\mathrm{d}a \\ & + k_1 \left[\int_0^A f_1(0) e^{-\int_0^a (\lambda+\mu_1(s))\,\mathrm{d}s}\,\mathrm{d}a + f_2\right] \int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a,\tilde{U}) u_1(a)\,\mathrm{d}a. \end{aligned}$$

$$\begin{aligned} f_1(0) = {} & f_1(0) \int_0^A \tilde{\varphi}_1(a,\tilde{U}) e^{-\int_0^a (\lambda+\mu_1(s))\,\mathrm{d}s}\,\mathrm{d}a \\ & + \left[k_1 f_1(0) \int_0^A e^{-\int_0^a (\lambda+\mu_1(s))\,\mathrm{d}s}\,\mathrm{d}a + k_1 f_2\right] \int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a,\tilde{U}) u_1(a)\,\mathrm{d}a. \end{aligned}$$

$$\begin{aligned} f_1(0) = {} & f_1(0) \int_0^A \tilde{\varphi}_1(a,\tilde{U}) e^{-\lambda a} \pi_1(a)\,\mathrm{d}a \\ & + \left[k_1 f_1(0) \int_0^A e^{-\lambda a} \pi_1(a)\,\mathrm{d}a + k_1 f_2\right] \int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a,\tilde{U}) u_1(a)\,\mathrm{d}a. \end{aligned}$$

Let $H_1 = \displaystyle\int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a,\tilde{U}) u_1(a)\,\mathrm{d}a$.

$$f_1(0) = f_1(0) \int_0^A \tilde{\varphi}_1(a,\tilde{U}) e^{-\lambda a} \pi_1(a)\,\mathrm{d}a + \left[k_1 f_1(0) \int_0^A e^{-\lambda a} \pi_1(a)\,\mathrm{d}a + k_1 f_2\right] H_1.$$

$$f_1(0) = f_1(0) \int_0^A \tilde{\varphi}_1(a,\tilde{U}) e^{-\lambda a} \pi_1(a)\,\mathrm{d}a + k_1 f_1(0) H_1 \int_0^A e^{-\lambda a} \pi_1(a)\,\mathrm{d}a + k_1 f_2 H_1.$$

Let $K_1(a) = \tilde{\varphi}_1(a,\tilde{U}) \pi_1(a)$, so we obtain

$$f_1(0) = f_1(0) \int_0^A K_1(a) e^{-\lambda a}\,\mathrm{d}a + k_1 f_1(0) H_1 \int_0^A e^{-\lambda a} \pi_1(a)\,\mathrm{d}a + k_1 f_2 H_1. \tag{21}$$

The integral in the equation (21) can be extended by zero to infinity, this leads that $\hat{K}_1(\lambda) = \displaystyle\int_0^\infty K_1(a) e^{-\lambda a}\,\mathrm{d}a$, $\hat{\pi}_1(\lambda) = \displaystyle\int_0^\infty e^{-\lambda a} \pi_1(a)\,\mathrm{d}a$, denote respectively the Laplace transform of $K_1(a)$ and $\pi_1(a)$.
Then (21) becomes

$$f_1(0)[1 - (\hat{K}_1(\lambda) + k_1 H_1 \hat{\pi}_1(\lambda))] - k_1 H_1 f_2 = 0. \tag{22}$$

From (22), and the last equation of (20), we obtain the following system

$$\begin{cases} f_1(0)[1 - (\hat{K}_1(\lambda) + k_1 H_1 \hat{\pi}_1(\lambda))] - k_1 H_1 f_2 = 0 \\ (\lambda - A_2) f_2 - A_1 f_1(0) \hat{\pi}_1(\lambda) = 0. \end{cases} \tag{23}$$

System (23) is written in the following matrix $A(\lambda)$

$$\begin{pmatrix} 1-(\hat{K}_1(\lambda)+k_1H_1\hat{\pi}_1(\lambda)) & -k_1H_1 \\ -A_1\hat{\pi}_1(\lambda) & \lambda-A_2 \end{pmatrix} \begin{pmatrix} f_1(0) \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation corresponds to the steady states $(u_1(a), u_2^*)$ is given by

$$det(A(\lambda)) = \left[1-\left(\hat{K}_1(\lambda)+k_1H_1\hat{\pi}_1(\lambda)\right)\right](\lambda-A_2)-k_1A_1\hat{\pi}_1(\lambda)H_1=0. \tag{24}$$

Stability of the trivial steady state

For $u^0=(0,0)$ we have $H_1=0$ and $A_2=m-g_0$. Then (24) becomes

$$\left(1-\hat{K}_1(\lambda)\right)(\lambda-(m-g_0))=0,$$

which implies that $1=\hat{K}_1(\lambda)$ or $\lambda=m-g_0$.
To complete our study, we need the following proposition.

Proposition 2.

Let $K_1(a)\geq 0$.
If $\int_0^{\infty} K_1(a)\,\mathrm{d}a>1$ *then the equation* $1-\hat{K}_1(\lambda)=0$ *has a real positive solution* λ.
If $\int_0^{\infty} K_1(a)\,\mathrm{d}a<1$ *then the equation* $1-\hat{K}_1(\lambda)=0$ *has no complex solution* λ *with* $Re(\lambda)>0$.

In this case, we have $R_1=\int_0^{\infty} K_1(a)\,\mathrm{d}a$ Then we have the following result.

Proposition 3.

If $R_1>1$ *then the equation* $1-\hat{K}_1(\lambda)=0$ *has a real positive solution* λ.
If $R_1<1$ *then the equation* $1-\hat{K}_1(\lambda)=0$ *has no complex solution* λ *with* $Re(\lambda)>0$.

We have the following results.

Theorem 4.

If $R_1<1$ *and* $m<g_0$ *then the trivial steady state is L.A.S.*
If $R_1<1$ *and* $m>g_0$ *then the trivial steady state is unstable.*
If $R_1>1$ *and* $m<g_0$ *then the trivial steady state is unstable.*
If $R_1>1$ *and* $m>g_0$ *then the trivial steady state is unstable.*

Stability of the blast steady state

For $u^b=(0,\frac{b_2}{\alpha})$ we have $H_1=0$ and $A_2=-\frac{ng_0(m-g_0)}{m}$.
Then (24) becomes

$$\left(1-\hat{K}_1(\lambda)\right)\left(\lambda+\frac{ng_0(m-g_0)}{m}\right)=0.$$

Then we obtain
$1 = \hat{K}_1(\lambda)$ or $\lambda = -\dfrac{ng_0(m-g_0)}{m} < 0$ ($m - g_0 > 0$ is condition of existence of u^b).
In this case, we have

$$\hat{K}_1(0) = \frac{(\alpha\theta)^n}{(\alpha\theta)^n + k_1^n b_2^n} R_1.$$

Then we give the following proposition.

Proposition 5.

If $R_1 > \left(\dfrac{k_1}{\alpha\theta}\right)^n \left(\dfrac{m-g_0}{cg_0}\right) + 1$ *then the equation* $1 - \hat{K}_1(\lambda) = 0$ *has a real positive solution* λ.
If $R_1 < \left(\dfrac{k_1}{\alpha\theta}\right)^n \left(\dfrac{m-g_0}{cg_0}\right) + 1$ *then the equation* $1 - \hat{K}_1(\lambda) = 0$ *has no complex solution* λ *with* $Re(\lambda) > 0$.

Now we declare the stability theorem of the blast steady state .

Theorem 6. *Let* $m > g_0$
If $R_1 < \left(\dfrac{k_1}{\alpha\theta}\right)^n \left(\dfrac{m-g_0}{cg_0}\right) + 1$ *then the blast steady state is L.A.S.*
If $R_1 > \left(\dfrac{k_1}{\alpha\theta}\right)^n \left(\dfrac{m-g_0}{cg_0}\right) + 1$ *then the blast steady state is unstable.*

Stability of the non pathological steady state

For $u^{np} = (\dfrac{b_1}{k_1 L_1}\pi_1(a), 0)$, equation (24) becomes

$$\left(1 - (\hat{K}_1(\lambda) + k_1 H_1 \hat{\pi}_1(\lambda)\right)\left(\lambda - \left(\frac{mk_1^n}{k_1^n + cb_1^n} - g_0\right)\right) = 0.$$

Then, we obtain $\left(1 - (\hat{K}_1(\lambda) + k_1 H_1 \hat{\pi}_1(\lambda)\right) = 0$ or $\lambda = \dfrac{mk_1^n}{k_1^n + cb_1^n} - g_0$.
We have

$$\begin{aligned} H_1 &= \frac{b_1}{k_1 L_1} \int_0^A \frac{\partial \tilde{\varphi}_1}{\partial \tilde{U}}(a, \tilde{U}) \pi_1(a)\, \mathrm{d}a \\ &= \frac{b_1}{k_1 L_1} \frac{-n\theta^n b_1^{n-1}}{(\theta^n + (b_1^n)^2} \int_0^A \varphi_1(a)\pi_1(a)\, \mathrm{d}a. \end{aligned}$$

Then we have

$$H_1 = \frac{-n\theta^n b_1^n}{k_1 L_1 (\theta^n + b_1^n)^2} R_1.$$

Lemma 7. *Let* $R_1 > 1$.
For $S(\lambda) = \hat{K}_1(\lambda) + k_1 H_1 \hat{\pi}_1(\lambda)$, $\eta_1 = \min_{a\in[0,A]} \varphi_1(a)$ *and* $\eta_2 = \max_{a\in[0,A]} \varphi_1(a)$, *then* $S(\lambda) \geq 0$ *if and only if* $L_1 < \dfrac{n}{n\eta_2 - \eta_1}$.

On the other hand, we have $\dfrac{d}{d\lambda} S(\lambda) = -a \displaystyle\int_0^A e^{-\lambda a} \left(K_1(a) + k_1 H_1 \pi_1(a)\right) \mathrm{d}a.$
Then $S(\lambda)$ is strictly decreasing under the condition $L_1 < \dfrac{n}{n\eta_2 - \eta_1}$.
Moreover

$$\begin{aligned} S(0) &= \int_0^A K_1(a)\,\mathrm{d}a + k_1H_1\int_0^A \pi_1(a)\,\mathrm{d}a = \int_0^A K_1(a)\,\mathrm{d}a + k_1H_1L_1 \\ &= \int_0^A K_1(a)\,\mathrm{d}a + \frac{-n(R_1-1)}{R_1}. \end{aligned}$$

With $K_1(a) = \tilde{\varphi_1}(a,\tilde{U})\pi_1(a) = \tilde{\varphi_1}(a,b_1)\pi_1(a) = \dfrac{\theta^n\varphi_1(a)}{\theta^n + b_1^n}\pi_1(a)$, this gives us

$\displaystyle\int_0^A K_1(a)\,\mathrm{d}a = 1.$

Then $S(0) = 1 + \dfrac{-n(R_1-1)}{R_1} < 1$ if and only if $R_1 > 1$.

Moreover $\lim\limits_{\lambda\to\infty} S(\lambda) = 0$.

Now we give the following proposition.

Proposition 8.

If $R_1 > 1$, then the equation $\hat{K}_1(\lambda) + k_1H_1\hat{\pi}_1(\lambda) = 0$ has no complex solution λ with $Re(\lambda) > 0$.

We have the following results.

Theorem 9. *Let $R_1 > 1$ and $L_1 < \dfrac{n}{n\eta_2 - \eta_1}$.*

If $R_1 > (\dfrac{k_1}{\theta})^n b_2^n + 1$, then $u^{np}(a)$ is L.A.S.

If $R_1 < (\dfrac{k_1}{\theta})^n b_2^n + 1$, then $u^{np}(a)$ is unstable.

Stability of the chronic steady state

For $u^c = \left(\dfrac{\alpha b_1 - k_1b_2}{k_1L_1(\alpha-1)}\pi_1(a), \dfrac{k_1b_2 - b_1}{k_1(\alpha-1)}\right)$, equation (24) becomes

$$\left(1 - (\hat{K}_1(\lambda) + k_1H_1\hat{\pi}_1(\lambda))\right)(\lambda - A_2) - k_2A_1\hat{\pi}_1(\lambda)H_1 = 0. \tag{25}$$

In this case, we have $A_1 = \dfrac{-mncb_2^{n-1}(k_1b_2 - b_1)}{k_1(\alpha-1)(1+cb_2^n)^2}$, $A_2 = \dfrac{-mnc\alpha b_2^{n-1}(k_1b_2 - b_1)}{k_1(\alpha-1)(1+cb_2^n)^2}$.

We note that $A_2 = \alpha A_1$.

On the other hand, we have

$$\begin{aligned} H_1 &= \int_0^A \frac{\partial\tilde{\varphi}_1}{\partial\tilde{U}}(a,\tilde{U})u_1(a)\,\mathrm{d}a \\ &= \frac{\alpha b_1 - k_2b_2}{L_1(\alpha k_1 - k_2)}\int_0^A \frac{\partial\tilde{\varphi}_1}{\partial\tilde{U}}(a,\tilde{U})\pi_1(a)\,\mathrm{d}a \\ &= \frac{-n\theta^n b_1^{n-1}(\alpha b_1 - k_1b_2)}{L_1k_1(\theta^n + b_1^n)^2(\alpha-1)}R_1. \end{aligned}$$

Idea if we find at least one eigenvalue λ where $Re(\lambda) > 0$ then we deduce directly that $u^c(a)$ is unstable (we specify the condition of instability).

We have

$det(A)\mid_{\lambda=0} = \alpha A_1(\hat{K}_1(0) - 1) + \hat{\pi}_1(0)n^2mc\theta^n b_2^{n-1}b_1^{n-1}\dfrac{(k_1b_2 - b_1)(\alpha b_1 - k_1b_2)}{L_1k_1(\alpha-1)(1+cb_2^n)^2(\theta^n + b_1^n)^2}R_1.$

We want to show that $det(A)\mid_{\lambda=0} < 0$.

We have

$$\hat{K}_1(0) = \int_0^\infty K_1(a)\,\mathrm{d}a = \int_0^\infty \tilde{\varphi}_1(a,\tilde{U})\pi_1(a)\,\mathrm{d}a = \frac{\theta^n}{\theta^n + b_1^n}R_1 = \frac{\theta^n}{\theta^n + \theta^n(R_1-1)}R_1. \\ = 1.$$

Then $(\hat{K}_1(0)-1)=0$ implies that $\alpha A_1(\hat{K}_1(0)-1)=0$.
It remains for us to demonstrate that:
$\hat{\pi}_1(0)n^2 mc\theta^n b_2^{n-1}b_1^{n-1}\dfrac{(k_1b_2-b_1)(\alpha b_1-k_1b_2)}{L_1k_1(\alpha-1)(1+cb_2^n)^2(\theta^n+b_1^n)^2}R_1 < 0.$
We have $(k_1b_2-b_1)<0$ and $k_1(\alpha-1)<0$ (condition of positivity of u_2^* in $u^c(a)$).
We have also $(\alpha b_1-k_1b_2)<0$ (condition of positivity of $u_1(a)$ in $u^c(a)$).
Then $det(A)\mid_{\lambda=0}<0$, and $\lim\limits_{\lambda\to+\infty} det(A)=+\infty$, so there is at least one eigenvalue λ_1 with $Re(\lambda_1)>0$, $det(A)\mid_{\lambda=\lambda_1}=0$, we conclude that $u^c(a)$ is unstable.
The stability theorem is given us follows.

Theorem 10. *Let* $m>g_0$

If $\left(\dfrac{k_1}{\theta}\right)^n\left(\dfrac{m-g_0}{cg_0}\right)+1<R_1<\left(\dfrac{k_1}{\alpha\theta}\right)^n\left(\dfrac{m-g_0}{cg_0}\right)+1,$ *then* u^c *is unstable.*

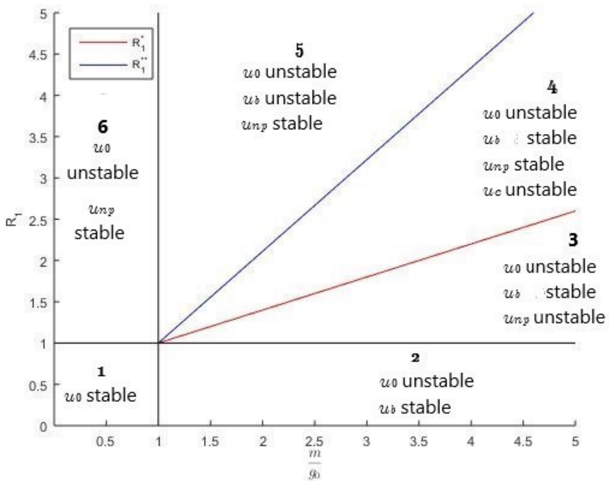

Figure 1: Existence and stability of the steady states case: $\alpha<1$ and $R_1^*<R_1<R_1^{**}$, with $R_1^*=(\dfrac{k_1}{\theta})^n(\dfrac{m-g_0}{cg_0})+1$, $R_1^{**}=(\dfrac{k_1}{\alpha\theta})^n(\dfrac{m-g_0}{cg_0})+1$

Numerical simulations

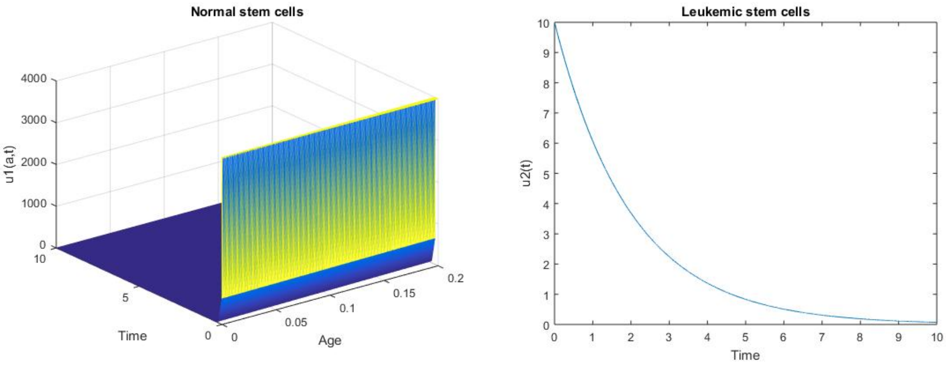

Figure 2: Simulations of normal and leukemic stem cells for the parameters: $\varphi_1(a) = 0.85 \times exp(-a)$, $\mu_1 = 0.005$, $\theta = 1.62 \times 10^8$, $n = 2$, $k_1 = 0.8$, $m = 0.005$, $\alpha = 0.9$, $g_0 = 0.5$, $c = 0.1$, case $R_1 < 1$, we are in the zone 1 figure (1), we have $R_1 = 0.1540$, then u^0 is stable, see (Th 3.1)

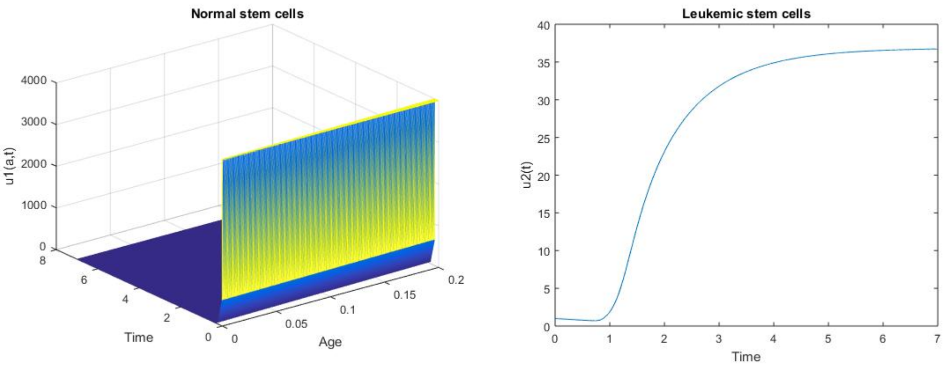

Figure 3: Simulations of normal and leukemic stem cells for the parameters: $\varphi_1(a) = 0.85 \times exp(-a)$, $\mu_1 = 0.005$, $\theta = 1.62 \times 10^8$, $n = 2$, $k_1 = 0.8$, $m = 6$, $\alpha = 0.9$, $g_0 = 0.5$, $c = 0.01$, case $R_1 < 1$, we are in the zone 2 figure (1), we have $R_1 = 0.1540$, then u^0 is unstable and u^b is stable, see (Th 3.1; 3.2)

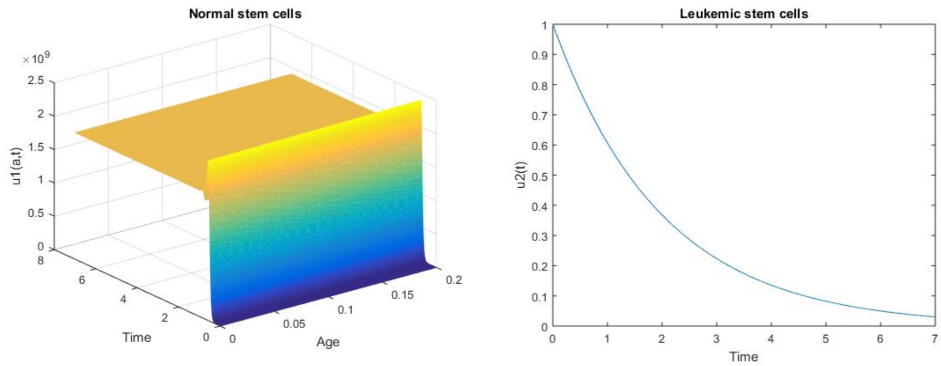

Figure 4: Simulations of normal and leukemic stem cells for the parameters: $\varphi_1(a) = 25 \times exp(-a)$, $\mu_1 = 0.005$, $\theta = 1.62 \times 10^8$, $n = 2$, $k_1 = 0.8$, $m = 6$, $\alpha = 0.9$, $g_0 = 0.5$, $c = 0.01$, case $R_1 > 1$, $R_1 > R_1^*$, we are in the zone 5 figure 1, we have $R_1 = 4.5295$, then u^0 is unstable, u^b is unstable and u^{np} is stable, see (Th 3.1; 3.2; 3.3)

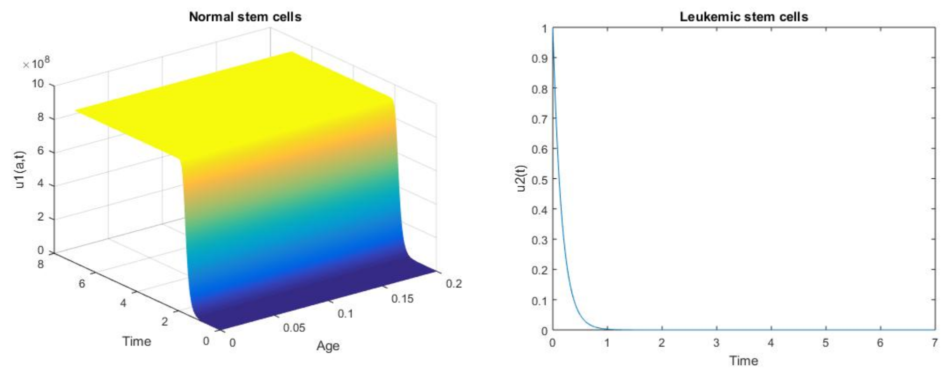

Figure 5: Simulations of normal and leukemic stem cells for the parameters: $\varphi_1(a) = 10 \times exp(-a)$, $\mu_1 = 0.005$, $\theta = 1.62 \times 10^8$, $n = 2$, $k_1 = 0.8$, $m = 0.5$, $\alpha = 0.9$, $g_0 = 6$, $c = 0.1$, case $R_1 > 1$, we are in the zone 6 figure (1), we have $R_1 = 1.8118$, then u^0 is unstable and u^{np} is stable, see (Th 3.1; 3.3)

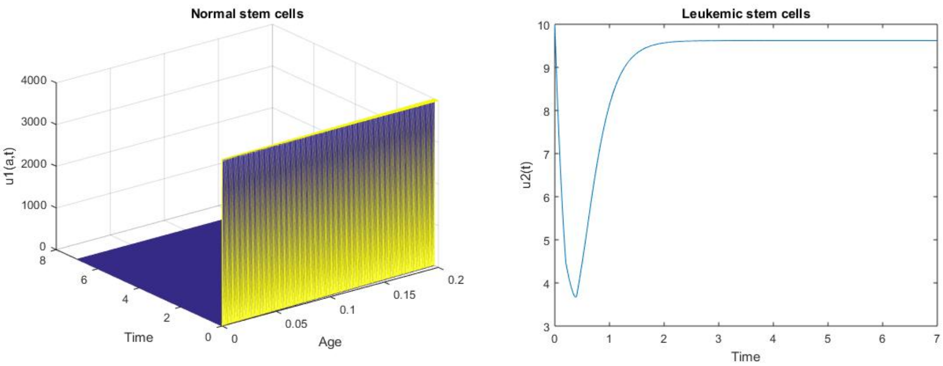

Figure 6: Simulations of normal and leukemic stem cells for the parameters: $\varphi_1(a) = 7.22 \times exp(-a)$, $\mu_1 = 0.005$, $\theta = 1$, $n = 2$, $k_1 = 0.8$, $m = 7$, $\alpha = 0.9$, $g_0 = 4$, $c = 0.01$, case $R_1 > 1$ and $R_1 < R_1^*$, we are in the zone 3 figure (1), we have $R_1 = 1.3081$, $R_1^* = 49$, then u^0 is unstable, u^b is stable, u^{np} is unstable, see (Th 3.1; 3.2; 3.3)

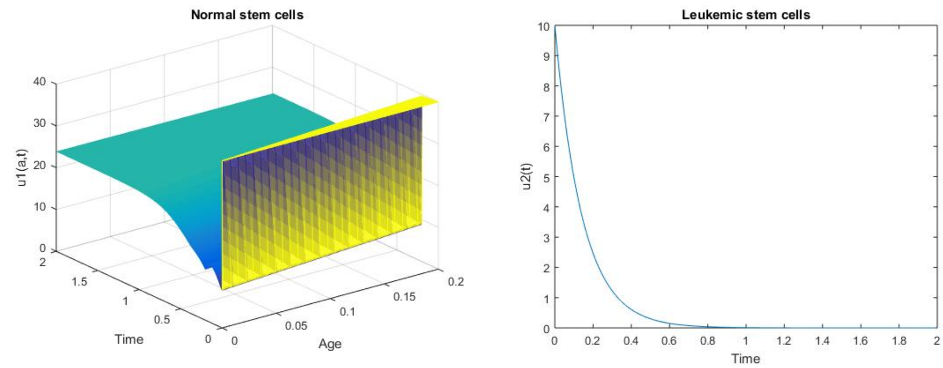

Figure 7: Simulations of normal and leukemic stem cells for the parameters: $\varphi_1(a) = 7.5 \times exp(-a)$, $\mu_1 = 0.05$, $\theta = 16$, $n = 2$, $k_1 = 2$, $m = 8$, $\alpha = 0.09$, $g_0 = 7$, $c = 0.09$, case $R_1^* < R_1 < R_1^{**}$, we are in the zone 4 figure 1, we have $R_1^* = 1.0248$, $R_1 = 4.0619$, $R_1^{**} = 1.3530$, then u^c is unstable, u^0 is unstable, u^b is stable, u^{np} is stable, see (Th 3.1; 3.2; 3.3; 3.4)

Conclusion

In this work, we have established an hybrid mathematical model of chronic myeloid leukemia inspired by the works in [2, 5, 7]. Our mathematical study was based on the research of the steady states and the demonstration of the stability that give us an idea about the behavior of our solutions.

The conditions of the stability of steady states were been given by the comparison of the parameters and the net reproduction rate R_1 of the model (3) (see Th 3.1, 3.2, 3.3, 3.4), these steady states are schematized in figure (1) which represent six zones of existence and stability of the steady states, we can notice easily that the steady states change there stability from one zone to an other.

In figure (1), if $m < g_0$ and the net reproduction rate $R_1 < 1$, we are in the zone (1) with a single steady state u^0 that is stable, while it becomes unstable in the others zones. The blast steady state exists in zones (2; 3; 4; 5) under the condition $m > g_0$,

and it's stable only in the case $R_1 < R_1^{**}$.
The non pathological steady state exists in the case $R_1 > 1$ (zones 3; 4; 5; 6), and it changes stability according to its own position compared to R_1^*. The last steady state is the chronic which is always unstable if $R_1^* < R_1 < R_1^{**}$ (see zone 4).
We finish our work by a numerical simulation (see Figure 2-7) which confirmed our mathematical results.
In the future work, we are going to treat a case of optimal control where we will add a treatment term in the model.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Frequency domain normalizations of EEG signals

Ayhan Savkliyildiz *[1] *and Yakup Irim* [2]

In this paper, commonly used frequency domain normalizations and their effects on electroencephalography (EEG) signal are examined. Additionally, a novel normalization method is introduced to decrease between subject variability in frequency domain. This method basically uses the affine transformation to warp the frequency domain. Alpha band peak frequency and it's near-harmonics are prominently active during eyes closed EEG recordings. However, these peak values can vary from person to person. Normalizing the frequency plane by mapping these peaks to fixed frequency positions on the frequency axis can reduce the variation caused by these oscillations. Thus, group analyzes can be evaluated in standardized time-frequency spectra.

2020 MSC: 15A04, 65T50, 65T60

KEYWORDS: Electroencephalography, Frequency domain, Normalization, Alpha band, Beta band

Introduction

Electroencephalography is a non-invasive tool and is used to detect electrical activity through electrodes placed on the scalp, but it cannot precisely pinpoint the exact location of neural activity within the brain. EEG signals can be attenuated (weakened) as they pass through the skull and scalp. This can also make it challenging to detect deep brain activity accurately. It provides relatively poor spatial resolution compared to other neuroimaging techniques like fMRI (Functional magnetic resonance imaging) or PET (Positron emission tomography). However, it is a powerful tool that is often used in clinics to detect the presence and location of functional disorders in the brain.

EEG signals are generally analyzed over five fundamental frequency bands. Among these bands, the alpha band (7-13Hz) is a band that is frequently investigated due to its dominance in brain neurodynamics [1, 3, 4, 6]. Especially in measurements taken with eyes closed, an increase in alpha band signals towards the back of the head (occipital lobe) is observed. This increase usually makes the Alpha band dominant over some other bands. Due to the predominance of this band, many studies have investigated its selectivity in different test setups and in different patient groups [1-7].

Research on alpha oscillations has repeatedly shown that the frequency of the alpha rhythm and the magnitude of the alpha band power decrease with age [3, 6]. According to a former suggestion [7], the beta peak frequency as observed during rest could be a second harmonic of alpha. On the other hand, Haegens and their colleague's study [5] found a positive correlation between alpha and beta peak frequencies during rest, suggesting that beta peak frequency as observed during rest in the parietal cortex could be a second harmonic of alpha. However, for the other conditions, there was no sign of correlation between alpha and beta peak frequencies, which argues against a simple harmonic relationship. The lack of beta modulation across conditions while alpha frequency increased suggests that they are largely independent. Since they are largely independent and decreased in frequency and magnitude with age, alpha-beta peak frequencies can be used in the normalization stages to reduce intra-subject variability.

Preliminaries

Power Spectral Density (PSD) main used in signal processing and can be defined equation below;

$$P(f) = \lim_{T\to\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} x(t) e^{-i2\pi ft}\, dt \right|^2 \tag{1}$$

where $P(f)$ represents the Power Spectral Density at frequency, $x(t)$ is the EEG signal in the time domain. T is the length of the time window over which the signal is analyzed, i represents the imaginary unit.

Normalization, in various contexts, refers to the process of bringing something into a standard or normalized state. It is commonly used in different fields, such as; statistics, database management, machine learning, signal processing. Signal processing normalizations are generally used in many fields of science when evaluating data gathered from different people or setups. It provides several advantages such as improved signal-to-noise ratios, consistent comparison, reduced distortion, increased robustness, and efficient feature extraction. On the contrary, the possibility of information loss and the increased computational requirements can emerge.

There are some essential normalization methods used in the frequency domain of EEG signals. Most of the time, normalization in the frequency domain is calculated over the amplitude. The normalization calculated over the amplitude are given below;

Min-Max normalization: Min-Max normalization scales the power of a signal to a specified range, often between 0 and 1, while preserving the relative relationships between the values.

$$P^*(f) = \frac{P(f) - min(P)}{max(P) - min(P)}. \tag{2}$$

Absolute normalization: This method involves expressing the power at each frequency as a proportion of the total power across all frequencies. This is often used to correct for variations in overall signal amplitude across different EEG recordings.

$$P^*(f) = \frac{P(f)}{\int P(f)}. \tag{3}$$

Relative power normalization: This method involves expressing the power at each frequency as a proportion of the power within a specific frequency band (e.g., the alpha band). This is often used to correct for variations in power within specific frequency bands across different individuals. This method have been commonly used through out the literature for comparison. Let P_R is the selected relative total band power.

$$P^*(f) = \frac{P(f) - min(P_R)}{max(P_R) - min(P_R)}. \tag{4}$$

Standardization, z-score normalization: Each data point in the signal is transformed by subtracting the mean of the signal and then dividing by the standard deviation. This ensures that the resulting signal has a mean of zero and a standard deviation of one, making it easier to compare and analyze different signals on a similar scale which helps in identifying patterns and anomalies within the data.

$$P^*(f) = \frac{P(f) - \mu}{\sigma}. \tag{5}$$

Proposed method

Standardized or normalized space is essential for investigation of group studies. Flowchart of proposed method given in Figure 1.

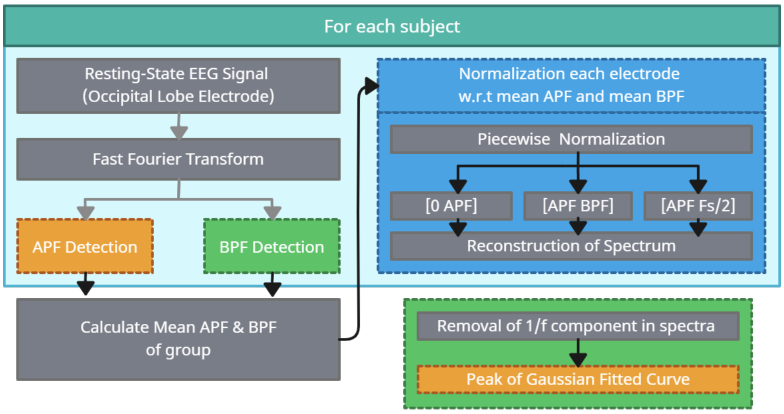

Figure 1: Flowchart of proposed spectral normalization method

Firstly, power spectral density of the resting state occipital lobe electrode's signal is calculated. Then, APF (Alpha Peak Frequency) is determined by fitting a gaussian curve between [8 13] Hz. The general equation for Gaussian curve fitting involves a function $f(x)$ with parameters,

$$f(x; A, \mu, \sigma) = A \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{6}$$

The goal of Gaussian curve fitting is to find the values of A, μ and σ that minimize the Least Mean Squared Error (LMSE) between the observed data and the fitted curve.

On the other hand, BPF (Beta Peak Frequency) is not as prominent as APF because of the decay component in the power spectrum. Therefore, the 1/f component in the spectrum must be subtracted to accurately detect the BPF. This component can be constructed with a suitable polynomial curves calculated over the spectrum excluding alpha and beta bands. Polynomial regression fits a polynomial function to the data is given by Equation 7.

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n. \tag{7}$$

Then, fitted polynomial curve can be substracted from the PSD of the signal. This way it will be much easier to detect the second bumb on PSD signal. Then, similar to alpha, gaussian fitted curve can be used to detect BPF.

Afterwards, mean of these peak values is calculated across subjects. Then, each subject's PSD signal divided into three part along the frequency axis such as [0, APF], [APF, BPF] and [BPF, fs/2], where fs is the sampling frequency. Each part of the subjects then normalized using linearly shrinking or strecthing of the PSD signal to groups mean frequencies respectively (i.e [0, APF] to [0, APF_{mean}]). This piecewise normalization can provide standardized spectrum for group analysis. Since APF and BPF will be the same for each subject, the subtraction and addition of PSD signals might show localized differences in other bands.

Conclusion and discussion

Brain oscillations of humans are similar, however, oscillations cannot be the same due to genetic and environmental factors. Also, age is an another factor that effects the oscillations (i.e. cognitive, sensory etc. [3, 5, 6]). These factors may effect the eventual frequency spectrum. Normalization of APF with its near-harmonics can be used to reduce these factors to localize active frequencies and electrodes efficiently. In our next study, it is planned to test the proposed method on EEG data recorded with eyes closed.

Additionally, wavelet-based detection of prominent peaks rather than power spectral density might have a significant impact for this proposed method. Because different types of wavelets affect the prominent peaks of the spectrum [2].

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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A new approach to deriving identities for the dual Bernstein basis functions

Ayse Yilmaz Ceylan

The aim of this talk is to derive a formula of the generating functions of dual Bernstein basis functions which has many applications in computer graphics and related areas. Especially, dual Bernstein functions and Bernstein functions are compared by an example in this study.

2020 MSC: 05A15, 65D17

KEYWORDS: Bernstein basis functions, Dual functions

Acknowledgments

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Compositions of integers whose each part is smaller than a positive integers

Busra Al [*1] *and Mustafa Alkan* [2]

In this study, we interested in the compostions of integers. Let n be positive integer and we define the set

$$P_n = \left\{(\alpha_1, \alpha_2, ..., \alpha_t) : \ \alpha_1 + \alpha_2 + ... + \alpha_t = n, \quad \alpha_i, t \in \mathbb{Z}^+\right\}.$$

In fact, P_n is the set of a compositions of an integer n and then two action on the set are defined. The decomposition of the composition sets of a positive integer has been examined by using set theory.
We define the set $P_{n,a}$ by restricted the compositions of the positive integers n. Finally, we obtain $Q_{n,a}$ by applying the product function we defined for the elements of this set.

2020 MSC: 05A15, 05A17, 05A18, 11B39, 11B99

Keywords: Compositions of an integer, Partitions of an integer, Restriction compositions, The generating function of an integer

Introduction

Partition theory is an important area of additive number theory, a subject concerning the representation of integers as sum of other integers. Recently, many researcher have written many publication on the theory of partition of a number because the rich history of partition of a number has been gone back to not only to very famous mathematicians Leonard Euler, but also Jacobi and also Indian mathematicians S. Ramanujan and English mathematicians G. H. Hardy.

Partition of a positive integer is expressing that number as the sum of positive integers. For a positive integer n, the partition function to be studied is the number of ways n can be written as a sum of positive integers n.

Partition are divided into compositions and partitions. The commutative of the sums in partition is not important, while the non-commutative of the sums in composition is important. While there are many studies in the literature about partitions in partition theory, there is only formula information about compositions that reports their number.

The total number of compositions of n is, in fact, given by 2^{n-1}. This is easily proved as follows:

The compositions of n can be classified under two heads:

1) Those in which the first part is 1; and

2) Those in which the first part is > 1.

Removing 1 (the first part), from each partition of n of the first kind, we obtain all the different partitions of $(n-1)$, each once. Reducing by 1 the first part in the partitions of the second kind, we again get all the partitions of $(n-1)$ as before. Hence, the number of partitions of n, is twice the number of partitions of $(n-1)$ and the result follows readily by induction [6].

Main results

From [2], we recall the composition of an integer n whose summands less than the fix integer a. For positive integers n, a, we use the notation the subset $P_{n,a}$ of P_n for the biggest summand a of composition of n i.e.

$$P_{n,\alpha} = \{(x_1, x_2, ..., x_t) : x_1 + ... + x_t = n, \text{ and } x_i \leq \alpha \text{ for all } i\}.$$

Now our aim is to find out the Multiplication function of $P_{n,\alpha}$ and also investigate for the generating function of this function. Therefore we provide some relations between the new number sequences and well known number sequences.

We defined the new action

$$1 \circledcirc (y_1, \ ..., y_t) = (1 + y_1, y_2, ..., y_t),$$

where $(y_1, \ ..., y_t) \in P_{n,\alpha} - \{(\alpha, x_1, x_2, ..., x_t) \in P_{n,\alpha}\}$. Then we use the notion $1 \circledcirc P_{n,\alpha}$ for the set of the element generated by the above operation, i.e.:

$$1 \circledcirc P_{n,\alpha} = \{(1 + x_1, x_2, ..., x_t) : x_1 + ... + x_t = n, \quad x_1 < \alpha \text{ and } x_i \leq \alpha \text{ for all } i\}.$$

We defined second operation with the a composition $\alpha = (\alpha_1, \alpha_2, ..., \alpha_t)$ of integer n;

$$(1 \odot \alpha) = (1, \alpha_1, \alpha_2, ..., \alpha_t).$$

Theorem 1 (*cf.* [2]). *For positive integers n, α, the set $P_{n+1,\alpha}$ is disjoint union of the sets $(1 \odot P_{n,\alpha})$ and $(1 \circledcirc P_{n,\alpha})$, i.e.*

$$P_{n+1,\alpha} = (1 \odot P_{n,\alpha}) \cup (1 \circledcirc P_{n,\alpha}). \tag{1}$$

Proof. By the definitions of the sets, it is clear that thes sets $(1 \odot P_{n,\alpha})$ and $(1 \circledcirc P_{n,\alpha})$ is disjoint.

Let $y \in P_{n,\alpha}$ then both $1 \odot y$ and $1 \circledcirc y \in P_{n+1,\alpha}$ and so

$$(1 \odot P_{n,\alpha}) \cup (1 \circledcirc P_{n,\alpha}) \subseteq P_{n+1,\alpha}.$$

Let $x = (x_1, ..., x_t) \in P_{n+1,\alpha}$ where $x_1 + ... + x_t = n+1$, and $x_i \leq \alpha$, for $i \in \{1, ..., t\}$. We have two cases:

i) If $x_1 \neq 1$ then we set $\gamma_1 = x_1 - 1$, $\gamma_i = x_i$, $i \in \{2, ..., t\}$ and it follows that $\gamma_1 < \alpha$ and $\gamma_1 + \gamma_2 + ... + \gamma_t = n$. This measn that $\gamma = (\gamma_1, \gamma_2, ..., \gamma_t) \in P_{n,\alpha}$ and we get $x = 1 \circledcirc (\gamma_1, \gamma_2, ..., \gamma_t) \in (1 \circledcirc P_{n,\alpha})$.

ii) If $x_1 = 1$ then we set $\gamma_{i-1} = x_i$, $i \in \{2, ..., t\}$ and so $\gamma_1 + \gamma_2 + ... + \gamma_{t-1} = n$. It follows that $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{m-1}) \in P_{n,\alpha}$ and thus $x = 1 \odot \gamma \in (1 \odot P_{n,\alpha})$.

Theorefore, $P_{n+1,\alpha} = (1 \circledcirc P_{n,\alpha}) \cup (1 \odot P_{n,\alpha})$ and so it complete the proof. □

Multiplication function of $P_{n,a}$

For positive integers n, α,we define the following function

$$Q_{n,\alpha} = \sum_{\substack{\alpha_1 + ... + \alpha_m = n \\ \alpha_i \leq \alpha}} \alpha_1.\alpha_2.....\alpha_m.$$

We may assume that $Q_{n,\alpha} = 0$ for non-positive integers n. It is clear that $Q_{n,\alpha} = Q_n = f_{2n}$ when $n \leq \alpha$.

Proposition 2. *For an intergers* n, α *with* $1 < n$, *we have that*

$$Q_{n+1,\alpha} = Q_{n,\alpha} + \sum_{i=1}^{\alpha-1} (1+i)\, Q_{n-i,\alpha}.$$

Proof. By Theorem , we get that the the recurrence

$$P_{n+1,\alpha} = (1 \odot P_{n,\alpha}) \cup (1 \circledcirc P_{n,\alpha})\,. \tag{2}$$

Then

$$\begin{aligned}
Q(P_{n+1,\alpha}) &= Q_{n+1,\alpha} = \left(\sum_{a\in P_{n,\alpha}} 1.\overbrace{\beta_1.\beta_2...\beta_m}^{\overline{\alpha}}\right) + \left(\sum_{\substack{\beta_1+\beta_2+...+\beta_m=n\\ \beta_2,..,\beta_m\leq\alpha,\beta_1<\alpha}} (1+\beta_1).\beta_2...\beta_m\right)\\
&= Q_{n,\alpha} + \sum_{i=1}^{\alpha-1}\sum_{\substack{\beta_2+...+\beta_m=n-i\\ \beta_2,..,\beta_m\leq\alpha}} \beta_2...\beta_m + \sum_{i=1}^{\alpha-1} i \sum_{\substack{\beta_2+...+\beta_m=n-i\\ \beta_2,..,\beta_m\leq\alpha}} \beta_2...\beta_m
\end{aligned}$$

$$\begin{aligned}
&= Q_{n,\alpha} + \sum_{i=1}^{\alpha-1} Q_{n-i,\alpha} + \sum_{i=1}^{\alpha-1} i\; Q_{n-i,\alpha}\\
&= Q_{n,\alpha} + \sum_{i=1}^{\alpha-1} (1+i\ )\, Q_{n-i,\alpha} \quad \text{(where } \alpha < n\text{)}.
\end{aligned}$$

□

Theorem 3. *For intergers* n, a, *we have the generating function*

$$M(\alpha,x) = \frac{Q_{1,\alpha}x + \sum_{n=2}^{\alpha}\left(Q_{n,\alpha} - Q_{n-1,\alpha} - \sum_{i=0}^{n-2}(n-i)\,Q_{i,\alpha}\right)x^n}{\left(1 - \sum\limits_{i=1}^{\alpha} ix^i\right)}.$$

Proof. Let $M(\alpha,x) = \sum_{n=1} Q_{n,\alpha}x^n$.

$$\begin{aligned}
M(\alpha,x) &= \sum_{n=1}^{\alpha-1} Q_{n,\alpha}x^n + x\sum_{n=\alpha} Q_{n+1,\alpha}x^n\\
&= \sum_{n=1}^{\alpha} Q_{n,\alpha}x^n + x\sum_{n=\alpha}\left(Q_{n,\alpha} + \sum_{i=1}^{\alpha-1}(1+i\ )\,Q_{n-i,\alpha}\right)x^n\\
&= \sum_{n=1}^{\alpha} Q_{n,\alpha}x^n - x\sum_{n=1}^{\alpha-1} Q_{n,\alpha}x^n + \left(x\sum_{n=1}^{\alpha-1} Q_{n,\alpha}x^n + x\sum_{n=\alpha} Q_{n,\alpha}x^n\right)\\
&\quad + x\sum_{n=\alpha}\sum_{i=1}^{\alpha-1}(1+i\ )\,Q_{n-i,\alpha}x^n\\
&= \sum_{n=1}^{\alpha} Q_{n,\alpha}x^n - x\sum_{n=1}^{\alpha-1} Q_{n,\alpha}x^n + xM(\alpha) +\\
&\quad \sum_{i=1}^{\alpha-1}\left((1+i)\,x^{i+1}M(\alpha) - (1+i)\,x^{i+1}\sum_{n=0}^{\alpha-i-1} Q_{n,\alpha}x^n\right)\\
&= \sum_{n=1}^{\alpha-1} Q_{n,\alpha}x^n\,(1-x) + Q_\alpha x^\alpha + \sum_{i=1}^{\alpha} ix^iM(\alpha) - \sum_{i=2}^{\alpha} ix^i \sum_{n=0}^{\alpha-i} Q_{n,\alpha}x^n.
\end{aligned}$$

Then we get that

$$M(\alpha,x)=\frac{\sum_{n=1}^{\alpha-1}Q_{n,\alpha}x^{n}\left(1-x\right)+Q_{\alpha,\alpha}x^{\alpha}-\sum\limits_{i=1}^{\alpha-1}\left(\left(1+i\right)x^{i+1}\sum_{n=0}^{\alpha-i-1}\ Q_{n,\alpha}x^{n}\right)}{\left(1-\sum\limits_{i=1}^{\alpha}ix^{i}\right)}.$$

If we rearrange the denominator, we have that

$$\begin{aligned}&\sum_{n=1}^{\alpha-1}Q_{n,\alpha}x^{n}\left(1-x\right)+Q_{\alpha,\alpha}x^{\alpha}-\sum_{i=1}^{\alpha-1}\left(\left(1+i\right)x^{i+1}\sum_{n=0}^{\alpha-i-1}Q_{n,\alpha}x^{n}\right)\\=\ &Q_{1,\alpha}t+\sum_{n=2}^{\alpha}\left(Q_{n,\alpha}-Q_{n-1,\alpha}-\sum_{i=0}^{n-2}\left(n-i\right)Q_{i,\alpha}\right)x^{n}\end{aligned}$$

and so we obtain the generating. □

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Direct and inverse discrete Sturm-Liouville problems with finitely many points of discontinuity

Bayram Bala

In this work, the second order difference equation which is equivalent to the Sturm-Liouville problem with discontinuous points at p interior points is investigated. We solved direct and inverse spectral problems according to the Generalized spectral function (GSF). Then the formulas are drived for the reconstruction of the coefficients matrix of the difference equation system.

2020 MSC: 39A12, 34A55, 34L15

KEYWORDS: Difference equation, Inverse problems, Generalized spectral function

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Regression analysis of chemical compound of Hurma region spring water

Berna Cayhan [*1] *and Mustafa Gurhan Yalcin* [2]

Studies on chemical and physical anomalies have gained importance in natural environment conditions such as water, soil and rock around the world. It is important to investigate heavy metals and physical properties in spring waters, which are thought to be used in drinking water. Furthermore, the interpretation of multivariate statistical analyzes during these studies is gaining importance day by day. The aim of the study is to perform and interpret regression analyzes on chemical contents and physical properties in Pinarbasi Spring, located in Hurma Region, west of Antalya city center. Regression analysis was performed with the SPSS 22 computer program in the chemical analysis data of the water samples taken in two different months. According to the regression analysis results, Durbin-Watson= 1.881, $R^2 = 0.997$ and $Sig. = 0.000$

2020 MSC: 6206, 62H10, 62P30, 86A32

KEYWORDS: Spring water, Regression analysis, Heavy metal, Hurma, Antalya gulf

Introduction

Scientifically, statistical studies in soil, rock and industry are very important (*cf.* [1, 2, 4], [7]-[11]). From past to present, drinking and utility water has gained importance throughout the world. Previously, the presence and absence of water was discussed, but in recent years, the pollution and cleanliness of the waters, which have decreased, has started to be discussed. The importance of scientific studies in waters has increased day by day. For this reason, the source of the water, its environment and the existing natural conditions have started to be examined more. There are many scientific studies on spring waters (*cf.* [3, 6, 5, 12]). The importance of these researches increases even more because these waters, which flow uncontrollably in their natural environment, need to be used by living beings. No scientific studies have been found on the Pinarbasi water source, which flows uncontrollably in its natural environment and is used by humans as drinking water. Therefore, there is a need for more scientific studies on Pinarbasi water source. The aim of the study is to perform a regression analysis on the physical properties and chemical contents of Pinarbasi water source in the Hurma region of Antalya. In addition, the results obtained will be interpreted.

Materials and method

By taking precaution of artificial contamination, the samples taken from Pinarbasi water source were placed in special bottles and their physical and chemical analyzes were made. Chemical analyzes were performed using the X-Ray fluorescence (XRF) method. Regression analysis was performed on the obtained physical and chemical analysis data.

Results and discussion

Regression analysis was performed on the results of physical and chemical analysis of water samples taken from Pinarbasi water source. When “Model Summary” is examined, the R Square (R^2) value is calculated as 0.997. This value showed that the data from the study area was sufficient. The “Durbin-Watson" value was calculated as 1.881 and the value was around +2 (Table 1). The range required for the model is between −3 and +3. When “ANOVA” is examined in statistical analysis, “Sig.” value was calculated as 0.000. There is no error value for the data.

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	0,999	0,997	0,997	5,49000	1,881

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	263122,667	3	87707,556	2910,000	0,000[b]
	Residual	783,641	26	30,140		
	Total	263906,308	29			

Table 1: Model summary and ANOVA

In the analysis of Residuals Statistics, “Std. Residual” is between −3.404 and +2.821 (Table 2). It ranged from −3.5 to +2.8. However, “Cook’s Distance” was 0 to 9, 369. We can say that there are extreme values among these analysis results and that extreme values affect these analysis values.

Residuals Statistics

	Minimum	Maximum	Mean	Std. Deviation	N
Predicted Value	-0,2078	445,0039	39,4285	95,25332	30
Std. Predicted Value	-0,416	4,258	0,000	1,000	30
Standard Error of Predicted Value	1,061	4,781	1,674	1,122	30
Adjusted Predicted Value	-0,2172	447,8734	39,0336	95,45284	30
Residual	-18,68670	15,48598	0,00000	5,19828	30
Std. Residual	-3,404	2,821	0,000	0,947	30
Stud. Residual	-4,451	4,458	0,023	1,272	30
Deleted Residual	-31,95609	38,67960	0,39487	9,82099	30
Stud. Deleted Residual	-8,947	9,005	0,025	2,410	30
Mahal. Distance	0,117	21,029	2,900	6,011	30
Cook's Distance	0,000	7,441	0,383	1,479	30
Centered Leverage Value	0,004	0,725	0,100	0,207	30

Table 2: Residuals statistics

To calculate the regression coefficient, it is necessary to find the one-unit change in the independent variable. This change may be decreasing or increasing. In such a case, it is necessary to find out how many times this value will change in the dependent variable. In the linear model established for the data, parameter estimation was made. A type of residue often used to identify outliers in a regression model is used here, known as "standardized residual: regression standardized residual". The model formed between this regression and the "erased regression residual: regression deleted residual" was formed. 91% of "regression standardized residual" can be explained by "regression deleted residual". The resulting equation is presented below.

$$\begin{aligned} Y &= 0,04 + 0,09X \\ R^2 &= 0,914 \text{ (Figure 1).} \end{aligned}$$

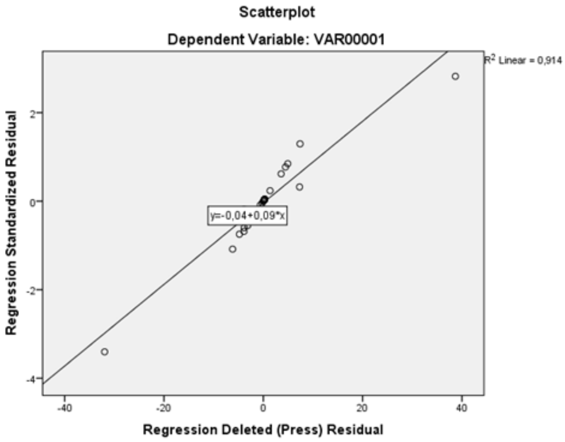

Figure 1: Scatterplot Diagram

Conclusion

According to the Regression analysis of the physical and chemical analysis results of Parbaşwater source, it was understood that statistically significant results were obtained by looking at Durbin-Watson= 1.881, $R^2 = 0.997$ and $Sig. = 0.000$ values. It can be stated that the established model is a statistically significant model. The association of "Regression deleted residual" and "Regression standardized residual" is derived as simple linear equation as $Y = 0,04 + 0,09X, R^2 = 0,914$.

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A Boubaker matrix method for solving high order differential equation with constant coefficients

Suayip Yuzbasi *[1] *and Beyza Cetin* [2]

In this article, a method is developed for solve high order differential equations. This method is created on the basis of Boubaker polynomials. As the first step of the method the default solution form and its derivatives are written in matrix forms. In addition, the function of the independent variable in the equation is written in the Maclaurin series. By generating matrix relations, the equation is transformed into a linear algebraic system. As a result of solving this system, an approximate solution is found.

2020 MSC: 33F05, 34K28, 35G05, 40C05, 74G15

KEYWORDS: High-order differential equation, Boubaker polynomials, Matrix method

Introduction

Higher order differential equations have a wide range of uses. Recently, various articles have been written using these equations such as computational mechanics [10], physics, which pioneered useful studies in many fields [9], medicine [13] and engineering that frequently uses equation modeling [1]. Due to this wide usage, many methods are available in the literature such as Galerkin method [12], Kernel method [6], Bessel polynomial method [16], Pell-Lucas collocation method [17] and pseudospectral method [3]. The m-order differential equation can be written as [5]

$$\sum_{k=0}^{m} P_k y^{(k)}(r) = f(r) \qquad a \le r \le b \tag{1}$$

with initial conditions

$$\sum_{k=0}^{m} (a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b)) = \lambda_i, \qquad i = 0, 1, ..., m-1. \tag{2}$$

This research presents a boubaker method for higher order differential equations. For this purpose, the continuation of the article will be as follows: establishing matrix relationships, creating the method, presenting the effectiveness of the method with applications, and finally, comments about the method will be presented in the results section.

The operational matrices

Matrix relations will be given in this section. For this purpose, it is possible to examine the equation in two parts. In two subheadings, these established matrix relationships will be introduced. For this aid Eq. (1) can be written in the form

$$\Gamma(r) = f(r) \tag{3}$$

where

$$\Gamma(r) = \sum_{k=0}^{m} P_k y^{(k)}(r). \tag{4}$$

The matrix form of $\Gamma(r)$

Boubaker polynomials are one of the orthogonal polynomials defined by [7]

$$B_w(\eta) = \sum_{j=0}^{[|w/2|]} (-1)^j (\frac{w-4j}{i-j}) \binom{w-j}{w} y^{w-2j} \tag{5}$$

and reiteration relation of the Boubaker polynomials defined as [14]

$$B_w(\eta) = tB_{w-1}(\eta) - B_{w-2}(\eta), \qquad w \geq 3.$$

We can achieve the approximate solution based on the Boubaker polynomials by means of [15]

$$y(r) \approxeq y_N(r) = \sum_{n=0}^{N} a_n B(r). \tag{6}$$

The matrix form expressed as,

$$y(r) \approxeq y_N(r) = \sum_{n=0}^{N} a_n B(r) = \mathbf{B}(r)\mathbf{A} \tag{7}$$

where

$$\mathbf{B}(r) = \begin{bmatrix} B_0(r) & B_1(r) & B_2(r) & \dots & B_N(r) \end{bmatrix} \quad \text{and}$$
$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_N \end{bmatrix}.$$

Then the vector $\mathbf{B}(r)$ composed Boubaker polynomials can be expressed as

$$\mathbf{B}(r) = \mathbf{X}(r)\mathbf{R}, \tag{8}$$

where

$$\mathbf{X}(r) = \begin{bmatrix} 1 & r & r^2 & \dots & r^N \end{bmatrix}$$

and if N is odd

$$\mathbf{R}^T = \begin{bmatrix} \xi_{0,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi_{1,0} & 0 & \dots & 0 & 0 \\ \xi_{2,1} & 0 & \xi_{2,0} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{N-1,\frac{N-1}{2}} & 0 & \xi_{N-1,\frac{N-3}{2}} & \dots & \xi_{N-1,0} & 0 \\ 0 & \xi_{N,\frac{N-1}{2}} & 0 & \dots & 0 & \xi_{N,0} \end{bmatrix}$$

and if N is even

$$\mathbf{R}^T = \begin{bmatrix} \xi_{0,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi_{1,0} & 0 & \dots & 0 & 0 \\ \xi_{2,1} & 0 & \xi_{2,0} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \xi_{N-1,\frac{N-2}{2}} & 0 & \dots & \xi_{N-1,0} & 0 \\ \xi_{N,\frac{N}{2}} & 0 & \xi_{N,\frac{N-2}{2}} & \dots & 0 & \xi_{N,0} \end{bmatrix}.$$

The relation between the terms of the Boubaker polynomial is as follows [8],

$$\xi_{n,k} = (-1)^k \left(\frac{n-4k}{k!}\right) \prod_{j=k+1}^{2k-1} (n-j), \qquad n,k = 0,1,...,N.$$

By putting Eq. (8) into the Eq. (7), we get

$$y(r) \approxeq y_N(r) = \mathbf{X}(r)\mathbf{R}\mathbf{A}. \tag{9}$$

Using these matrix definitions, we can derive derivative relations for Boubaker polynomials as

$$\begin{aligned} \mathbf{B}(r) &= \mathbf{X}(r)\mathbf{R}, \\ \mathbf{B}'(r) &= \mathbf{X}(r)\mathbf{S}\mathbf{R}, \\ \mathbf{B}''(r) &= \mathbf{X}(r)\mathbf{S}^2\mathbf{R}, \\ &\vdots \\ \mathbf{B}^{(k)}(r) &= \mathbf{X}(r)\mathbf{S}^{(k)}\mathbf{R}, \end{aligned}$$

where

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Hence, the kth-order derivative of Eq. (24) is obtained as

$$y^{(k)}(r) \approxeq y_N^{(k)}(r) = \mathbf{X}(r)\mathbf{S}^{(k)}\mathbf{R}\mathbf{A}. \tag{10}$$

The matrix form of $f(r)$

Now we can make the construction matrix form of $f(r)$ on the right side of equation. By using the Maclaurin expansion , $f(r)$ can be typeble as [11]

$$[f(r)] \approxeq \sum_{k=0}^{N} f_n r^k = \mathbf{X}(r)\mathbf{F} \tag{11}$$

where

$$f_n = \frac{f^{(n)}(0)}{n!} \qquad \text{and} \qquad \mathbf{F} = [f_0 \quad f_1 \quad f_2 \dots \quad f_N]^T. \tag{12}$$

Because of the relation (12), we can briefly define as follow:

$$[f(r)] \approxeq \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} r^k = \mathbf{X}(r)\mathbf{F}_\gamma \tag{13}$$

where

$$\mathbf{F}_\gamma = \left[\frac{f^{(0)}(0)}{0!} \quad \frac{f^{(1)}(0)}{1!} \quad \frac{f^{(2)}(0)}{2!} \dots \quad \frac{f^{(N)}(0)}{N!}\right]. \tag{14}$$

From here we get the relation as

$$[f(r)] = \mathbf{X}(r)\mathbf{F}_\gamma. \tag{15}$$

The matrix form of initial conditions

The initial conditions (2) of Eq. (1) as

$$\sum_{k=0}^{m}(a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b))=\lambda_i, \qquad i=0,1,...,m-1. \tag{16}$$

The correlation of (10) obtained above is used, the we get

$$\left[\sum_{k=0}^{m-1} a_{ik}\mathbf{X}(a)\mathbf{S}^k+b_{ik}\mathbf{X}(b)\mathbf{S}^k\right]\mathbf{R}=\left[\lambda_i\right], \qquad i=0,1. \tag{17}$$

Boubaker collocation method

The previous section describes the matrix relationships required to construct the fundamental matrix equation. In this section, the basic matrix equation of Eq. (1) will be construct with the help of these relations. For this purpose, if Eqs. (10)-(15) are substituted in Eq. (1), we reached

$$\sum_{k=0}^{m} P_k\mathbf{X}(r)\mathbf{S}^{(k)}\mathbf{R}\mathbf{A}=\mathbf{X}(r)\mathbf{F}_\gamma \qquad a\leq r\leq b \tag{18}$$

and from here it is possible to obtain the following equation as

$$\left[\sum_{k=0}^{m} P_k\mathbf{S}^{(k)}\mathbf{R}\right]\mathbf{A}=\mathbf{F}_\gamma. \tag{19}$$

This equation can be written briefly

$$\mathbf{WA}=\mathbf{G} \quad \text{or} \quad [\mathbf{W};\mathbf{G}] \tag{20}$$

where

$$\mathbf{W}=\sum_{k=0}^{m} P_k\mathbf{S}^{(k)}\mathbf{R} \quad \text{and} \quad \mathbf{G}=\mathbf{F}_\gamma. \tag{21}$$

In addition, we obtain rows by using the matrix form we created for the initial conditions. These lines are obtained as

$$\mathbf{U}_i\mathbf{A}=[\lambda_i] \qquad \text{or} \qquad [\mathbf{U}_i;\lambda_i] \tag{22}$$

where

$$\mathbf{U}_i=\sum_{k=0}^{m-1} a_{ik}\mathbf{X}(a)\mathbf{S}^k\mathbf{R}=\left[u_{i0} \quad u_{i1}\ldots u_{iN}\right], \qquad i=0,1...m-1. \tag{23}$$

Substituting the given initial conditions into the (23), we get

$$[\upsilon_{00} \quad \upsilon_{01}\ldots\upsilon_{0N}]=\lambda_0$$
$$[\upsilon_{10} \quad \upsilon_{11}\ldots\upsilon_{1N}]=\lambda_1.$$
$$[\upsilon_{20} \quad \upsilon_{21}\ldots\upsilon_{2N}]=\lambda_2.$$
$$\vdots$$

$$[u_{m-1,0} \quad u_{m-1,1} \dots u_{m-1,N}] = \lambda_{m-1}.$$

If the rows here are replaced with the rows of the linear algebraic system, the augmented matrix is obtained as

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{F}}\right] = \begin{bmatrix} \omega_{00} & \omega_{01} & \dots & \omega_{0N} & ; & f(\eta_0) \\ \omega_{10} & \omega_{11} & \dots & \omega_{1N} & ; & f(\eta_1) \\ \omega_{20} & \omega_{21} & \dots & \omega_{2N} & ; & f(\eta_2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{N-m,0} & \omega_{N-m,1} & \dots & \omega_{N-m,N} & ; & f(\eta_{N-(m-1)}) \\ \upsilon_{00} & \upsilon_{01} & \dots & \upsilon_{0N} & ; & \lambda_0 \\ \upsilon_{10} & \upsilon_{11} & \dots & \upsilon_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \upsilon_{m-1,0} & \upsilon_{m-1,1} & \dots & \upsilon_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}.$$

By solving this system, the unknown coefficients matrix **A** is obtained. The obtained coefficients $a_1, a_2, ..., a_N$ are replaced into the

$$y_N(r) = \mathbf{X}(r)\mathbf{R}\mathbf{A} \tag{24}$$

and thus the approximate solution sought is found.

Numerical experiment

Example 1. *Let us examine 4th-order differential equation given by the following;*

$$y^{(4)} - 5y = -5r^5 + 25r^2 + 120r, \quad y(0) = 0, y'(0) = 0, y''(0) = -10, y'''(0) = 0. \tag{25}$$

The exact solution for this problem is $r^5 - 5r^2$. Now, applying the method we developed in previous section, an approximate solution is found for $N = 5$. The fundemantal matrix equation of the Eq. (25) *is*

$$\left\{\mathbf{S}^4\mathbf{R} - 5\mathbf{R}\right\}\mathbf{A} = \mathbf{F}. \tag{26}$$

A linear algebraic system is created to find the approximate solution. For this aim, we can determine the matrices X and F as follows:

$$\mathbf{W} = \begin{bmatrix} -53 & 0 & 0 & 0 & 24 & 0 \\ 0 & 365 & 0 & -120 & 0 & 120 \\ -10 & 0 & -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & -5 & 0 & 0 \\ 10 & 0 & 0 & 0 & -5 & 0 \\ 0 & 15 & 0 & 5 & 0 & -5 \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} 0 \\ 120 \\ 25 \\ 0 \\ 0 \\ -5 \end{bmatrix}. \tag{27}$$

The linear algebraic system is calculated as

$$[\mathbf{W};\mathbf{F}] = \begin{bmatrix} -53 & 0 & 0 & 0 & 24 & 0 & ; & 0 \\ 0 & 365 & 0 & -120 & 0 & 120 & ; & 120 \\ -10 & 0 & -5 & 0 & 0 & 0 & ; & 25 \\ 0 & -5 & 0 & -5 & 0 & 0 & ; & 0 \\ 10 & 0 & 0 & 0 & -5 & 0 & ; & 0 \\ 0 & 15 & 0 & 5 & 0 & -5 & ; & -5 \end{bmatrix}. \tag{28}$$

Also the rows obtained from the conditions are as follows

$$[\mathbf{U_0}; \lambda_\mathbf{0}] = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad 0]$$

$$[\mathbf{U_1}; \lambda_\mathbf{1}] = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad ; \quad 0]$$

$$[\mathbf{U_2}; \lambda_\mathbf{2}] = [4 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \quad ; -10]$$

and

$$[\mathbf{U_3}; \lambda_\mathbf{3}] = [0 \quad 6 \quad 0 \quad 6 \quad 0 \quad 0 \quad ; \quad 0].$$

By substituting these rows in the last lines of the linear algebraic system, the augmented system is obtained as

$$[\mathbf{W}; \mathbf{F}] = \begin{bmatrix} -53 & 0 & 0 & 0 & 24 & 0 & ; & 0 \\ 0 & 365 & 0 & -120 & 0 & 120 & ; & 120 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & ; & 0 \\ 4 & 0 & 2 & 0 & 0 & 0 & ; & -10 \\ 0 & 6 & 0 & 6 & 0 & 0 & ; & 0 \end{bmatrix}. \quad (29)$$

By solving this system, matrix **A** is found as

$$\mathbf{A} = [0 \quad 0 \quad -5 \quad 0 \quad 0 \quad 1]^T.$$

If this matrix is written in Eq. (24), the approximate solution sought is found as $y_5(r) = r^5 - 5r^2$ which is an exact solution.

Example 2. Let us examine the second problem given as

$$y''' + 2y'' - 3y' = 17e^{2r} + 10re^{2r}, \quad y(0) = 0, \quad y'(0) = 2, \quad y''(0) = 5. \quad (30)$$

The exact solution for this problem is $e^r + re^{2r}$. If the method is applied for $N = 5$, 9 and 13 for this problem, the following approximate solutions are obtained.

$y_5(r) = 0.675r^5 + 1.375r^4 + 2.1667r^3 + 2.5r^2 + 2r + 1,$

$y_9(r) = 0.0064r^9 + 0.0254r^8 + 0.8909r^7 + 0.268r^6 + 0.675r^5 + 1.375r^4 + 2.1667r^3 + 2.5r^2 + 2r + 1,$

$y_{13}(r) = 0.00008r^{13} + 0.00005r^{12} + 0.00028r^{11} + 00.0014r^{10} + 0.0064r^9 + 0.0254r^8 + 0.8909r^7 + 0.268r^6 + 0.675r^5 + 1.375r^4 + 2.1667r^3 + 2.5r^2 + 2r + 1.$

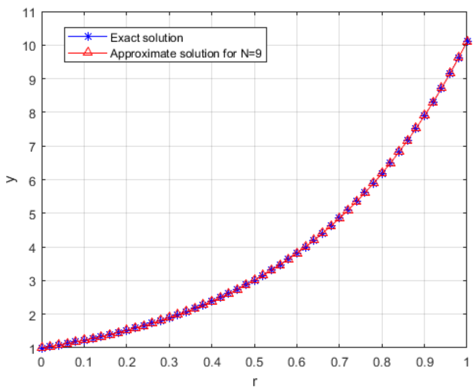

Figure 1: Comparising approximate solution for N=9 with exact solution Example 2

Figure 1 shows the comparison of the exact solution and the obtained approximate solution for N=9. The values obtained for different N values are also shown in table 1. It is assimilated that the values obtained here get closer to the accuracy as N gets larger.

r	Exact Solution $e^r + re^{2r}$	Approximate solution $N = 5$	Approximate solution $N = 9$	Approximate solution $N = 13$
0,0	1	1	1	1
0,2	1.519767697688424	1.519749336000000	1.519767700207417	1.519767700357922
0,4	2.382041069038257	2.380778688000000	2.382040929818083	2.382041090549212
0,6	3.814188954032437	3.798688072000000	3.814179347221860	3.814189026982588
0,8	6.187966868008560	6.093717504000000	6.187787219827163	6.187966985199483
1,0	10.10733792738969	9.716666999999999	10.10558349199999	10.10733677408999

Table 1: Comparising exact solution with approximate solution of Eq. (30)

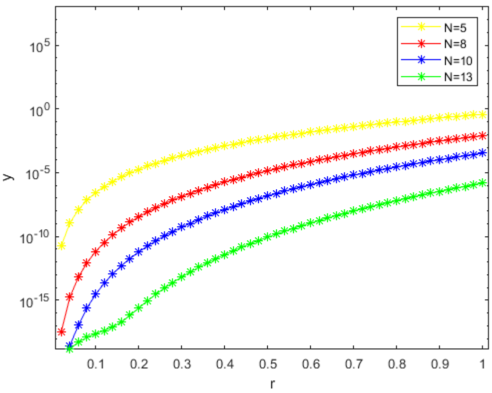

Figure 2: Comparising approximate solution for N=5.8.10 and 13 of Example 2

Figure 2 consists of absolute error functions. Here, the graphs obtained for different N values are compared. As can be seen here, the error function obtained for N=8 gave better results than N=5. This is also true for N=8, 10 and 13. The larger the N value, the smaller errors are obtained.

Example 3. Let us examine the problem given as

$$9y'' + y' = 0, \quad y(0) = 0, y'(0) = 1. \tag{31}$$

The exact solution for this problem is $9 - 9e^{-r/9}$. This equation is a 2nd order homogeneous constant coefficient ordinary differential equation.

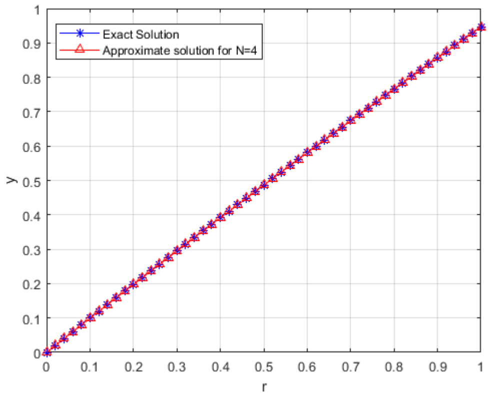

Figure 3: Comparising approximate solution for N=9 with exact solution Example 3

r	Present Method $N = 4$	Present Method $N = 6$	Present Method $N = 8$	Present Method $N = 9$	Present Method $N = 12$
0,0	0	0	0	0	0
0,2	4.0494e-10	4.7666e-15	1.0389e-18	1.0062e-18	1.0062e-18
0,4	1.2910e-08	6.0831e-13	2.0810e-17	4.0286e-18	4.1029e-18
0,6	9.7678e-08	1.0364e-11	6.5027e-16	5.1315e-18	9.4066e-18
0,8	4.1010e-07	7.7435e-11	8.5335e-15	5.8728e-17	1.7034e-17
1,0	1.2469e-06	3.6822e-10	6.3340e-14	6.7708e-16	2.7103e-17

Table 2: Comparising absolute errors of Eq. (31)

Figure 3 shows the comparison of the exact solution and the obtained approximate solution for N=4. Absolute error functions show the effectiveness of the method. Absolute errors for different N values 0are given in Table 2.

Conclusion

In this research, a new solution method for higher order differential equations is presented. This method is based on Boubaker polynomials. In the method established by creating matrix relations, functions that can be opened to the maclouren series were used. In addition, various examples are presented for this created method. These examples consist of both homogeneous and inhomogeneous differential equations. The effectiveness of the method is demonstrated with the help of the absolute error functions obtained in these examples. In the first example, a problem with a polynomial solution is considered and in this example, the exact solution is reached by using this method. For Example 2 in Table 1, it is commented that the higher the N value, the closer to the exact solution. In the third example, a homogeneous problem is considered. As can be seen in these examples, remarkable results have been obtained.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Wavelet-based analysis of the effect of electrode position on the EMG spectrum

Omer H. Colak [1], *Buket Simsek* [*2] *and Ovunc Polat* [3]

Surface Electromyography (EMG) is a technique used to measure the electrical activity of muscles. Electrode position is one of the most important factors in EMG signal measurement. While the correct electrode position helps to best record muscle activity, incorrectly placed electrodes can lead to misleading results. In this study, it is aimed to determine how the effect of electrode position on the EMG amplitude and spectrum changes with Wavelet-Based Analysis.

2020 MSC: 42C40, 34L16

KEYWORDS: Muscle activity, Electrode position, Wavelet transform, Surface electromyography(sEMG)

Introduction

Surface Electromyography (EMG) is a technique used to measure the electrical activity of muscles. The EMG signal is described as one of the best bioelectrical signals that can be detected from the external surface of the body (skin surface), conveying the electrical activities of muscles during contraction or relaxation (*cf.* [2, 5]). The electrode placement is one of the most crucial factors in EMG signal measurement. In EMG, the placement of electrodes on the skin surface plays a significant role in increasing the sensitivity and accuracy of the EMG signal, minimizing the activation of other muscles, and influencing diagnosis and treatment planning. Proper electrode positioning assists in capturing muscle activity optimally, whereas incorrectly placed electrodes can lead to misleading results (*cf.* [3]).

Wavelet analysis is one of the methods used to examine EMG signals in the time-frequency domain. Continuous wavelet transform is used to understand how the frequency components of the signal change over time. This analysis method helps to understand the changes in muscle activity over time and the changes that occur at different frequencies (*cf.* [6]).

In the context of this scope, the objective of the conducted study is to determine how the effect of electrode placement on the EMG spectrum changes using wavelet transform. The amplitude and spectral variations of EMG signals measured with electrodes positioned at different locations on the same muscle of the lower extremity have been analyzed using Continuous Wavelet Transform (CWT), and the differences have been evaluated.

Materials and methods

In this study, measurements were taken from the lower extremity to determine how the electrode placement affects the amplitude and spectrum of EMG signals. Tibialis muscle, which is the strong muscle in the lower leg muscle group, which will create high muscle activation, was preferred for the measurement. A study was conducted by recording sEMG (surface electromyography) from an individual. Since the Tibialis muscle is active during ankle dorsiflexion movement, the individual was instructed to perform ankle dorsiflexion movement (*cf.* [7]). The recordings were taken as three repetitions with four seconds of contraction and four seconds of rest. Simultaneous sEMG measurements were recorded from three different points on the muscle (muscle beginning, muscle mid, muscle end) in the standard position. Measurements Multi-channel sEMG recordings were taken at Akdeniz University Neuroscience Laboratory. For sEMG recordings, bioelectrical amplifiers and ADInstrument's PowerLab 35/8-35/16 data acquisition systems were used, with 9 mm bipolar electroencephalography (EEG) electrodes.

The amplitude and spectral changes of the recorded EMG signals were analyzed by CWT (Continuous Wavelet Transform). The Continuous Wavelet Transform (CWT) is a mathematical tool used for analyzing the frequency content of signals or time-series data in a localized manner. Complex wavelets are often used for CWT analysis as they allow the separation of phase and amplitude components associated with the signal (*cf.* [1]).

The formula for the CWT of a signal $f(t)$ with respect to a wavelet function $\psi(t)$ is given by,

$$CWT(a,b) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{a}} \psi^* \left(\frac{t-b}{a} \right) dt$$

where, a is the scale parameter, controlling the width of the wavelet function. b is the translation parameter, shifting the wavelet function along the time axis. $\psi^*(t)$ is the complex conjugate of the mother wavelet $\psi(t)$ (*cf.* [4]).

Results

Measurements taken from three different regions of the Tibialis muscle, namely middle, the beginning and the end point, have been subjected to CWT analysis, and the resulting scalograms are presented in Figure 1, Figure 2 and Figure 3.

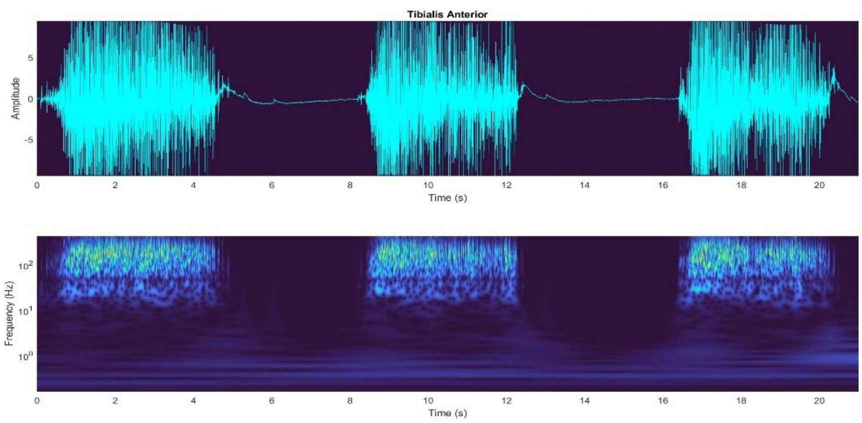

Figure 1: Scalogram of the signal received from the midpoint of the muscle

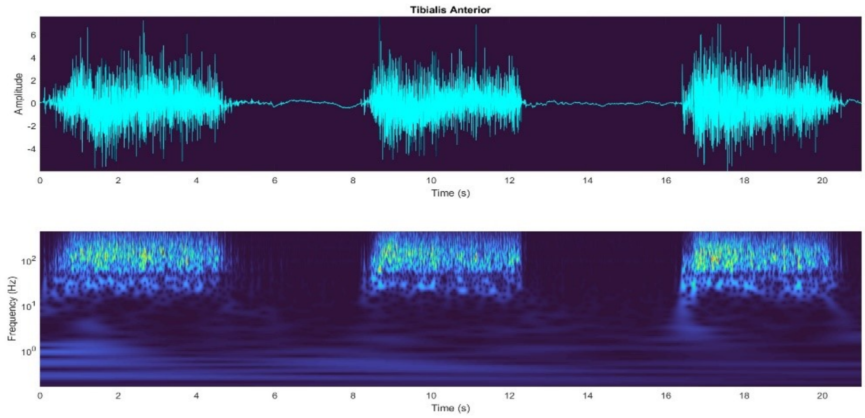

Figure 2: Scalogram of the signal received from the origin of the muscle

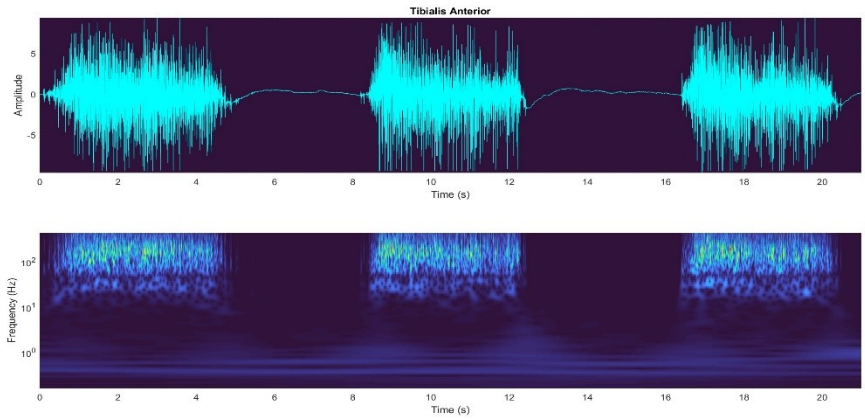

Figure 3: Scalogram of the signal received from the endpoint of the muscle

In the provided figures, it has been observed that the amplitude of the signal taken from the midpoint of the muscle is higher compared to the amplitude at the muscle origin, indicating a more intense reflection of muscle activation. The amplitude value at the muscle endpoint is observed to be similar to the amplitude of the signal taken from the midpoint of the muscle. When examining spectral changes, a consistent spectrum independent of muscle position is observed. At the beginning of the movement, there is a stronger activation, and similar patterns independent of muscle location are also detected here.

Conclusion

As a result of the findings, it has been evaluated that the sEMG measurement made from the midpoint of the muscle better reflects the muscle activation in terms of amplitude, but the spectral differences are very low when the spectral changes of the recordings taken from different points of the muscle are examined. At the point of generalizing this result, the study continues with multiple records. It will be possible to achieve more general results with an increase in the number of participants and a variety of motor movements, especially in muscle groups of different sizes.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Investigation of the effect of supply methods on fresh fruit and vegetable product purchase prices

Caner Atakoglu *[1] *and Yavuz Tascioglu* [2]

Today, due to the rapid increase in food prices on a global scale, agricultural activities also increase their importance for countries. In this context, national and international market chains, which bring all basic foodstuffs to the consumer, play an active role in the process. In Turkey, which is one of the important agricultural countries in the world, fresh fruit and vegetables are in demand in chain markets with fruit and vegetable aisles on a national scale as well as the demand it sees in its production and export. In this study, it will be examined the fresh fruit and vegetable supply methods of the national markets and the factors that are effective in the formation of product purchase prices. There are studies on supply chain and its methods in the literature. In the fresh fruit and vegetable category, supply and price processes will be examined in the dimension of chain markets. As a result of the study, the supply processes and price formations of chain markets that have a large market share in fresh fruit and vegetables will be examined in detail and contribute to the literature.

2020 MSC: 62H10, 62H05

KEYWORDS: Fresh fruits and vegetables, Retail chain market, Supply, Price

Introduction

The agricultural sector has a great importance in the country's economy. The vast majority of agricultural products are utilized through the supply of raw materials to industries, exports and domestic market sales. After the input market in the fresh vegetable and fruit supply chain, which is one of the sub-branches of agriculture; It is possible to make a classification as producers (producer associations), intermediaries (traders, brokers, etc.) and retailers (chain markets, etc.). Within the scope of this study, the supply methods of fresh fruit and vegetables in retail chain markets and the factors affecting the purchase prices in the procurement process will be revealed. It is one of the most important sub-branches of fresh fruit and vegetable agriculture. Fresh fruits and vegetables constitute an important part of the agricultural added value in the world and in Turkey. Considering the supply process, the fruits and vegetables grown as a result of production activities; It is expected that their organic structure will be delivered to the consumer without deteriorating or undergoing any process that will change their structure. This process includes processing, packaging, supply, delivery to retailers or directly to the consumer. Considering the actors in the process, a large part of the products meet with retailers (chain markets) through intermediaries such as collectors, brokers, brokers, forwarders, traders, and in the last step, they are offered to the consumer. Therefore, it is of great importance to investigate the fresh fruit and vegetable supply processes of the retailers, which have a large share in the delivery of fresh fruit and vegetables to the final consumer, and the factors affecting the prices within the scope of the study. The fact that fresh fruits and vegetables sold in a highly competitive environment change hands more than once until they reach the retailers (chain market) cause the main producer not to reach the desired income, while causing the consumer's purchase prices to increase. Some studies in the literature on the subject are summarized as follows; [1] studied producer-exporter supply relations by identifying 3 fruits and vegetables that play an important role in exports with the example of Antalya province in his doctoral thesis study in 2022. As a result of the study, it has been determined that the most important suppliers of exporting companies are brokers. In his thesis study conducted in 2022, [2] revealed that the effects of supply chain performance components on price, quality, delivery reliability, innovation, time to market play an important role. [3] in a study they conducted in 2016, revealed Turkey's fresh fruit and vegetable production structure and marketing channel. How the production areas and the products produced are presented to the marketing channel and the relations between the actors in the marketing channel and the producers are revealed.

In his 2020 study, [4] reveals that it has common stakeholders to minimize the losses experienced during the supply processes in fresh fruit and vegetables and it will be easy to cooperate.

Materials and method

The product should be presented to the consumer at the right time, in sustainable quality and quantity. Suppliers play an important role in the supply chain. Orders taken according to customer demands should be supplied as soon as possible and submitted through appropriate packaging processes. At this stage, the processes of receiving the products from the manufacturers, packaging processes and transporting them with appropriate vehicles to be delivered to the customer should be adjusted correctly and systematically. Fresh fruit and vegetable supply methods of national markets directly affect product purchase prices. Parameters such as manufacturer, packaging and transportation that play a role in procurement processes are difficult and variable to estimate. In this context, this study will focus on important issues and parameters in the supply of fresh fruits and vegetables in national markets.

Conclusion

Nutrition, which is among the most basic needs of humanity, directly concerns the agricultural sector. The agricultural sector not only meets the food needs of humanity, but also provides input for the country's economy. Turkey is advantageous compared to other countries in terms of agricultural production and export due to its location in the world. By using the advantages of the Mediterranean region climate zone and agricultural soils rich in minerals, Turkey has a wide variety of agricultural products, especially fresh fruit and vegetable cultivation. The Mediterranean region is the region with the most abundant fresh fruit and vegetable cultivation in Turkey. Antalya region is the province where 90% of fresh fruit and vegetables are grown. In this context, it is of great importance to convey the fresh fruit and vegetables grown in Antalya to the consumer in the most healthy and protected condition. A quality and standard should be kept at a certain level without losing quality and taste in the delivery of the products to the consumer. At this stage, the supply chain until the products grown in agricultural areas are delivered to the consumer should be discussed in detail. The supply chain should be thought of as a network. These studies will be carried out for the first time for the selected region and will shed light on the methods of supplying fresh fruits and vegetables in national markets around the world and the examination/evaluation stages of the factors affecting product purchase prices.

Acknowledgments

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Some arithmetic properties of Bernoulli numbers of higher orders

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Let r be any positive integer. In this work, we revisite explicit and recurrence formulas satisfied by the Bernoulli numbers $B_n^{(r)}$ of higher order r. By using the unsigned Stirling numbers, we give them new forms and a clearer aspect. The essential part of this study, consists in generalize the arithmetic properties satisfied by the classical Bernoulli numbers to the numbers $B_n^{(r)}$. Among other things, we establish Kummer congruences for the numbers $\frac{B_n^{(r)}}{r\binom{n}{r}}$ and we construct a new family of Eisenstein series whose constant term is $\frac{(-1)^r}{2}\frac{B_k^{(r)}}{r\binom{k}{r}}$ and which verifies Kummer congruences. We give explicit formulas for the denominators of the numbers $(r-1)!B_n^{(r)}$, as well as a formula analogous to that of Clausen-von Staudt. We finish our work by constructing several multiples of the numerators of the numbers $\frac{B_n^{(r)}}{r\binom{n}{r}}$ and $(r-1)!B_n^{(r)}$.

2020 MSC: 11B68, 11F33, 11M32, 11M36

Keywords: Stirling number, Bernoulli number of higher order, Kummer congruence, p-integrality, Clausen and von Staudt formula, Numerator of Bernoulli number of higher order, Eisenstein series

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The maths of a photo induced hydrogel swimming robot: Nonsmooth forcing dynamics

Chen Xuan

Certain hydrogel is light sensitive since light provides heat and hydrogel shrinks under high temperature thus making the hydrogel photosensitive. When light shines on the hydrogel, the hydrogel shrinks and as light propagates through the hydrogel shrinks less thus bending towards the light. This hydrogel beam undergoes a vibration governed by a wave PDE with a second derivative in time and 4^{th} derivative in space coupled with a diffusion equation, a result of a competition between the elasticity of the beam and the photo induced bending. This photo induced vibration enables the hydrogel beam to swim in water. Based on dimensional energy analysis, we determine the stable vibration amplitude and construct phase diagrams for the increase and decrease of the oscillation amplitude, which are further confirmed experimentally. It is found that resonance can occur and damping plays an important role in determining the conditions for resonance. A mass-spring-damper ODE system subjected to a displacement dependent excitation force is developed to investigate the features in generalized self excited oscillating systems. The prototypical PDEs can be well understood by the above simplified ODE model. This work lays a solid foundation for understanding self excited oscillation and provides design guidelines for self sustainable soft robots. It also puts forth another interesting question of whether chaos is involved in future work.

2020 MSC: 35-XX, 35Qxx

Keywords: Dynamical system, Nonsmooth forcing, Beam vibration, Multiphysics, Diffusion

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Interval decision-making problems

Dmitriy Dolgy

The method of partial ordering of real intervals with the help of a numerical indicator is considered. Applications of the indicator for the reduction of various classes of interval decision-making problems to deterministic problems of linear and non-linear programming, multi-objective optimization, and non-cooperative games are described. The properties of reduced problems are established. Illustrating examples are considered.

2020 MSC: 34K35, 47N70, 93-XX

KEYWORDS: Indicator of interval inequality, Reduction of decision-making problem, Nonlinear programming problem, Linear programming problem, Controllability of a discrete system

Introduction

Many mathematical models used in the natural sciences, engineering, social and humanitarian fields can be considered as interval ones. One of the examples is physics, in which the phenomena of the surrounding world are described by dependencies with experimental or approximately calculated data. Other examples are found in computational mathematics, inverse problem theory, optimization methods, operations research, economics, sociology, and other fields of knowledge. The needs of theory and practice stimulated the emergence in the 1950s of a new mathematical discipline - interval analysis. Initially, the objects of study of interval analysis were problems of computational mathematics, and the main efforts were focused on obtaining two-sided (interval) estimates of solutions. Almost at the same time, decision-making problems began to be investigated: extreme, game, controlled. Their specificity is related with the need to use the preference relation. In the "ordinary" extreme problem, the preference relation is naturally given by a real objective function. Of the two points in its domain, in the one of which the function has a smaller (or larger) value is considered preferable. In the case of an interval-valued objective function, in order to select a preferred point, it is necessary to compare two intervals of its values for "less" (or "greater").

Interval problems can be treated as a parametric family of problems generated by all parameter values in admissible intervals. In the monograph [1], the solution of the interval problem is considered to be one "acceptable" solution for the entire family of problems. It is shown that finding an acceptable ("universal") solution is reduced to solving a regularized deterministic problem of the same type as the original interval problem.

Another approach employs comparison of intervals. In [2]-[4], the partial order on the set of real intervals is understood in the strong and weak sense. Strong comparison is defined for intervals without common interior points; all other intervals are considered incomparable. A weak comparison allows the intersection of intervals, while the "smaller" interval on the numerical line may be to the right of the "larger" interval.

The problem of partial ordering of intervals can be approached applying the methodology of other mathematical disciplines - probability theory, fuzzy set theory. They use real indicators of binary operations on sets - probabilities of random events, membership functions. By analogy, it is natural to set a partial order relation on the set of real intervals by defining a pairwise comparison of intervals using a numerical indicator of interval inequality [5, 6]. As a result, a formal basis appears for correct mathematical formulations of interval decision-making problems and development of methods for their solution. The present work is devoted to the presentation of these questions.

Thematically, the presentation of the material consists of three related parts. The first part provides the necessary information about the indicator of interval inequality. The second part shows the application of the indicator for the study of interval decision-making problems. The reductions of typical problems to similar deterministic problems are described, in which the target conditions and constraints can be meaningfully interpreted in terms of probability. In the third part linear interval discrete systems are considered. The questions of representation of solutions and non-negative controllability of the system to a given position are clarified.

The results of applying the approach to economic models are given in [7]-[11].

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Relations of exponential Euler spline associated with special numbers and polynomials

Damla Gun *[1] *and Yilmaz Simsek* [2]

In this presentation, using some special numbers and polynomials with their generating functions and the exponential Euler splines, some formulas and relations are given. With the aid of the relations between some special numbers and polynomials, and the B-spline is given. We also give some interesting results including the Frobenius-Euler numbers of higher order, the Stirling numbers of the second kind, the Changhee numbers, the Daehee numbers, the Eulerian polynomials, the modification exponential Euler-type splines and beta-type rational functions.

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KEYWORDS: Apostol-Euler numbers and polynomials, Stirling numbers, Exponential Euler splines, Beta-type rational functions, Generating functions

Introduction

Simsek [21, 23] constructed generating functions for the following finite and infinite sums of powers of (inverse) binomial coefficients:

$$y_6(v,b;\omega,p)=\sum_{k=0}^{b}\binom{b}{k}^{p}\frac{\omega^k k^v}{b!} \tag{1}$$

and

$$B_v(\beta;\omega,p)=\sum_{m=0}^{\infty}\frac{m^v\omega^m}{\binom{\beta}{m}^{p}}, \tag{2}$$

where $b,v,p\in\mathbb{N}_0$ ($\mathbb{N}_0=\mathbb{N}\cup\{0\}=\{0,1,2,\ldots\}$), $\omega,\beta\in\mathbb{R}$, set of real numbers (or $\mathbb{C}$, set of complex numbers):

$${}_pF_{p-1}\left(-b,-b,...,-b;1,1,...,1;(-1)^p\,\omega e^u\right)=\sum_{v=0}^{\infty}b!y_6(v,b;\omega,p)\frac{u^v}{v!} \tag{3}$$

and

$${}_{p+1}F_p\left(1,1...,1;-\beta,...,-\beta;(-1)^p\,\eta e^u\right)=\sum_{v=0}^{\infty}B_v(\beta;\eta,p)\frac{u^v}{v!}, \tag{4}$$

where $-\beta\notin\{0,-1,-2,-3,\ldots\}$ and also $|\eta|<1$ and ${}_kF_m\left(x_1,...,x_k;w_1,...,w_m;u\right)$ denotes the following well-known generalized hypergeometric series:

$${}_kF_m\left(x_1,...,x_k;w_1,...,w_m;u\right)=\sum_{v=0}^{\infty}\left(\frac{\prod_{j=1}^{k}\left(x_j\right)_v}{\prod_{j=1}^{m}\left(w_j\right)_v}\right)\frac{u^v}{v!}, \tag{5}$$

where the above series converges for all u if $k < m+1$, and for $|u| < 1$ if $k = m+1$. Assuming that all parameters have general values, real or complex, except for the w_j, $j = 1, 2, ..., m$ none of which is equal to zero or a negative integer. $(w)_v$ denotes the Pochhammer's symbol, defined by

$$(w)_v = \prod_{j=0}^{v-1} (w + j)$$

with $(w)_0 = 1$, $v \in \mathbb{N}$ and $w \in \mathbb{C}$.

In this presentation, we also need the following generating functions:

Let $\eta \in \mathbb{C}$ with $\eta \neq 1$. The Frobenius-Euler polynomials of order α, $H_v^{(\alpha)}(s;\eta)$, and the Frobenius-Euler numbers of order α, $H_v^{(\alpha)}(\eta)$, are defined by means of the following generating functions:

$$\left(\frac{1-\eta}{e^u - \eta}\right)^{\alpha} e^{us} = \sum_{v=0}^{\infty} H_v^{(\alpha)}(s;\eta) \frac{u^v}{v!} \tag{6}$$

and

$$\left(\frac{1-\eta}{e^u - \eta}\right)^{\alpha} = \sum_{v=0}^{\infty} H_v^{(\alpha)}(\eta) \frac{u^v}{v!}. \tag{7}$$

When $\alpha = 1$,

$$H_v(s;\eta) = H_v^{(1)}(s;\eta)$$

denotes the Frobenius-Euler polynomials (*cf.* [2, 10, 18, 24, 25]).

The Apostol-Bernoulli polynomials $\mathcal{B}_v(s;\eta)$ are defined by

$$\frac{ue^{su}}{\eta e^u - 1} = \sum_{v=0}^{\infty} \frac{\mathcal{B}_v(s;\eta)}{v!} u^v, \tag{8}$$

where $|u| < 2\pi$ when $\eta = 1$; $|u| < |\log(-\eta)|$ when $\eta \neq 1$, and also $1^{\alpha} = 1$ (*cf.* [1, 11, 12, 22, 26]). When $s = 0$,

$$\mathcal{B}_v(0;\eta) = \mathcal{B}_v(\eta)$$

denotes the Apostol-Bernoulli numbers (*cf.* [1, 22, 26]).

Let $\alpha, \eta \in \mathbb{C}$. The Apostol-Euler polynomials of order α are defined by

$$\frac{2^{\alpha} e^{su}}{(\eta e^u + 1)^{\alpha}} = \sum_{v=0}^{\infty} \mathcal{E}_v^{(\alpha)}(s;\eta) \frac{u^v}{v!}, \tag{9}$$

where $|u| < \pi$ when $\eta = 1$; $|u| < |\log(-\eta)|$ when $\eta \neq 1$, and also $1^{\alpha} = 1$. When $s = 0$,

$$\mathcal{E}_v^{(\alpha)}(0;\eta) = \mathcal{E}_v^{(\alpha)}(\eta)$$

denotes the Apostol-Euler numbers of order α. Substituting $\alpha = 1$ into the above equation, we have the Apostol-Euler numbers

$$\mathcal{E}_v(\eta) = \mathcal{E}_v^{(1)}(\eta)$$

(*cf.* [11, 12, 26]).

Let $j \in \mathbb{N}_0$. The Stirling numbers of the second kind $S_2(v,j)$ are defined by

$$\frac{(e^u - 1)^j}{j!} = \sum_{v=0}^{\infty} S_2(v,j) \frac{u^v}{v!} \tag{10}$$

(*cf.* [1]-[26]).

The Eulerian polynomials are defined by

$$\frac{1-\eta}{1-\eta e^{u(1-\eta)}} = \sum_{v=0}^{\infty} \mathcal{A}_v(\eta) \frac{u^v}{v!},$$

where

$$\mathcal{A}_v(\eta) = \sum_{d=1}^{v} \left\langle \begin{array}{c} v \\ d \end{array} \right\rangle \eta^d$$

with

$$\left\langle \begin{array}{c} v \\ d \end{array} \right\rangle = \sum_{c=0}^{d} (-1)^c \binom{v+1}{c} (1+d-c)^v$$

(*cf.* [4, 5, 13]).

Some results and comments on the exponential Euler spline

In [17] and [18], Schoenberg gave the following definition of the exponential Euler spline of degree n with knots at the integer v:

$$s_v(s;u) = \frac{H_v(s;u)}{H_v(0;u)},$$

where $0 \leq s < 1$,

$$s_v(s+1;u) = u s_v(s;u)$$

and

$$s_v(s;u) \in C^{v-1}(\mathbb{R}).$$

It is clear to see that for $s \in \mathbb{Z}$, $s_v(s;u) = u^s$.

By using (2), Simsek [23] gave

$$B_v(-1;-\eta,1) = \sum_{m=0}^{\infty} m^v \eta^m, \tag{11}$$

where $|\mu| < 1$. In [18], it is easy to see that

$$\sum_{m=0}^{\infty} m^v \eta^m = \frac{\mathcal{A}_v(\eta)}{(1-\eta)^{v+1}}, \tag{12}$$

where $|\eta| < 1$. Combining (11) with (12), we get the following corollary:

Corollary 1. *Let $v \in \mathbb{N}_0$. Then we have*

$$\mathcal{A}_v(\eta) = (1-\eta)^{v+1} B_v(-1;-\eta,1).$$

The well-known formula for the Apostol-Euler numbers and the Apostol-Bernoulli numbers is given as follows:

$$-\frac{v}{2}\mathcal{E}_{v-1}(-\eta) = \mathcal{B}_v(\eta) \tag{13}$$

(*cf.* [26]). Simsek [23] gave the following formula for the polynomial $\Pi_v(\eta)$:

$$\Pi_{v-1}(\eta) = \frac{-(\eta-1)^v}{\eta}\mathcal{B}_v(\eta). \tag{14}$$

Here note that the polynomials $\Pi_v(\eta)$ are modifications of the numbers $H_n^{(\alpha)}(\eta)$. Combining (13) with (14), we find the following formula for the polynomials $\Pi_v(\eta)$:

Theorem 2. *Let $v \in \mathbb{N}_0$. Then we have*

$$\Pi_v(\eta) = (v+1)\frac{(\eta-1)^{v+1}}{2\eta}\mathcal{E}_v(-\eta). \tag{15}$$

Relations among the B-spline, the series $B_v(\beta;\eta,p)$, the polynomial $\Pi_v(\eta)$ and the Eulerian polynomials are given as follows.

Let $\eta_+ := \{0,\eta\}$ with $\eta \in (-\infty,\infty)$. The B-spline is defined by

$$Q_{k+1}(\eta) = \frac{1}{k!}\eta_+^k - \frac{1}{k!}\binom{k+1}{1}(\eta-1)_+^k + \cdots + \frac{(-1)^{k+1}}{k!}(\eta-1-k)_+^k,$$

where $Q_{k+1}(\eta) > 0$ if $0 < \eta < k+1$, $Q_{k+1}(\eta) = 0$ elsewhere (*cf.* [17, 18]). Since

$$\Pi_k(\eta) = k!\sum_{j=0}^{k-1} Q_{k+1}(j+1)\eta^j \tag{16}$$

(*cf.* [18]), with the aid of (15) and (16), we get the following corollary:

Corollary 3. *Let $v \in \mathbb{N}_0$. Then we have*

$$\frac{v+1}{2\eta}(\eta-1)^{v+1}\mathcal{E}_v(-\eta) = v!\sum_{j=0}^{v-1} Q_{v+1}(j+1)\eta^j.$$

Combining (15) with the following well-known identity (*cf.* [23]):

$$H_v(s;\eta) = \sum_{j=0}^{v}\frac{\Pi_j(\eta)s^{v-j}}{(\eta-1)^j},$$

gives the following corollary:

Corollary 4. *Let $v \in \mathbb{N}_0$. Then we have*

$$H_v(s;\eta) = \frac{\eta-1}{2\eta}\sum_{j=0}^{v}(j+1)\,s^{v-j}\mathcal{E}_j(-\eta).$$

By using the Apostol-Euler polynomials of order α, the Stirling numbers and the Frobenius-Euler polynomials of order α, we obtain the following well-known equations:

$$\frac{2^\alpha}{(1+\eta)^\alpha}H_v^{(\alpha)}\left(s;-\frac{1}{\eta}\right) = \mathcal{E}_v^{(\alpha)}(s;\eta) \tag{17}$$

and

$$\mathcal{E}_v^{(\alpha)}(s;\eta) = 2^\alpha\sum_{k=1}^{v}\sum_{j=1}^{k}(-1)^j\binom{v}{k}\frac{\eta^j S_2(k,j)(\alpha)_j}{(1+\eta)^{\alpha+j}}s^{v-k} \tag{18}$$

(*cf.* [26]). Combining (17) with (18), we have

$$H_v^{(\alpha)}\left(s;-\frac{1}{\eta}\right)=\sum_{k=1}^{v}\sum_{j=1}^{k}(-1)^j\binom{v}{k}\frac{\eta^j S_2(k,j)(\alpha)_j}{(1+\eta)^j}s^{v-k}.$$

Replacing η by $-\frac{1}{\eta}$ in the previous equation, we have

$$H_v^{(\alpha)}(s;\eta)=\sum_{k=1}^{v}\sum_{j=1}^{k}\binom{v}{k}\frac{S_2(k,j)(\alpha)_j}{(\eta-1)^j}s^{v-k}$$

and also Simsek [19, 20] gave the beta-type rational functions as follows:

$$\mathfrak{M}_{j,m}(\eta)=\eta^j(\eta+1)^{m-j}. \tag{19}$$

Combining the above equations with (19), we get the following corollary:

Corollary 5. *Let $v\in\mathbb{N}_0$. Then we have*

$$H_v^{(\alpha)}\left(s;-\frac{1}{\eta}\right)=\sum_{k=1}^{v}\sum_{j=1}^{k}(-1)^j\binom{v}{k}S_2(k,j)(\alpha)_j\, s^{v-k}\mathfrak{M}_{j,0}(\eta)$$

and

$$H_v^{(\alpha)}(s;\eta)=\sum_{k=1}^{v}\sum_{j=1}^{k}\binom{v}{k}S_2(k,j)(\alpha)_j\, s^{v-k}\mathfrak{M}_{0,-j}(-\eta).$$

Combining the above relation with the following known formula given for a class of modified of exponential Euler-type splines of degree v with order α:

$$Y(s,\eta;v,\alpha)=\frac{H_v^{(\alpha)}(s;\eta)}{H_v^{(\alpha)}(\eta)}$$

(*cf.* [7]), we get the following theorem:

Theorem 6. *Let $v\in\mathbb{N}_0$. Then we have*

$$Y(s,\eta;v,\alpha)=\frac{1}{H_v^{(\alpha)}(\eta)}\sum_{k=1}^{v}\sum_{j=1}^{k}\binom{v}{k}S_2(k,j)(\alpha)_j\, s^{v-k}\mathfrak{M}_{0,-j}(-\eta).$$

Recall that

$$\mathcal{B}_v(\eta)=\frac{v\eta}{(\eta-1)^v}\sum_{s=0}^{v-1}(-1)^s s!\eta^{s-1}(\eta-1)^{v-1-s}S_2(v-1,s) \tag{20}$$

(*cf.* [1]).

Boyadzhiev [3] gave the relationship between the Eulerian polynomials and the Apostol-Bernoulli numbers as follows:

$$A_v(\eta)=-\frac{(1-\eta)^{v+1}}{v+1}\mathcal{B}_{v+1}(\eta) \tag{21}$$

(*cf.* [28]). Combining (14) with (21), we have

$$\Pi_v(\eta)=(-1)^{v+1}\frac{(v+1)}{\eta}A_v(\eta).$$

Combining (19) and (21) with (20), we get the following corollary:

Corollary 7. *Let $v \in \mathbb{N}_0$. Then we have*

$$A_v(-\eta) = \sum_{s=0}^{v} (-1)^s s! S_2(v,s) \mathfrak{M}_{s,v}(\eta).$$

Combining the above equation with

$$(-1)^s s! = (s+1) D_s$$

and

$$(-1)^s s! = 2^s Ch_s,$$

where D_s and Ch_s denote the Daehee numbers and the Changhee numbers, respectively (*cf.* [8, 9]), we arrive at the following theorem:

Theorem 8. *Let $v \in \mathbb{N}_0$. Then we have*

$$A_v(-\eta) = \sum_{s=0}^{v} (s+1) D_s S_2(v,s) \mathfrak{M}_{s,v}(\eta)$$

and

$$A_v(-\eta) = \sum_{s=0}^{v} 2^s Ch_s S_2(v,s) \mathfrak{M}_{s,v}(\eta).$$

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Stirling numbers associated with sequences of polynomials

Dae San Kim *[1] *and Taekyun Kim* [2]

The Stirling, degenerate Stirling, Lah and Gould-Hopper numbers all appear in the expansions of some sequence of polynomials in terms of falling factorials and vice versa. We generalize this to any sequence of polynomials $\mathbf{P} = \{p_n(x)\}_{n=0}^{\infty}$ satisfying $\deg p_n(x) = n$, $p_0(x) = 1$. The aim of this paper is to study the Stirling numbers of the second kind associated with any sequence of polynomials and of the first kind associated with any sequence of polynomials, in a unified and systematic way with the help of umbral calculus technique. The Stirling numbers associated with any sequence of polynomials enjoy orthogonality and inverse relations. This is illustrated with many examples which give rise to interesting orthogonality and inverse relations in each case. Furthermore, this is the key fact in the recent applications of Lah transforms to the fields of telecommunication and optics.

2020 MSC: 05A19, 05A40, 11B73, 11B8

Keywords: Stirling numbers of the first kind associated with sequence of polynomials, Stirling numbers of the second kind associated with sequence of polynomials, Eulerian polynomials associated with sequence of polynomials, Umbral calculus

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Some properties of the Faber polynomials

Dmitry Kruchinin *1 *and Vladimir Kruchinin* 2

In this paper we consider the Faber polynomials. We obtain explicit formulas for the Faber polynomials. Also we get integer properties of the Faber polynomials. We give examples of the Faber polynomials for the generating functions $x + exp(x^{-1})$ and $(3x - \sqrt{x^2 - 4x})/2$.

2020 MSC: 05A15, 11B83

KEYWORDS: Generating function, Faber polynomials, Explicit formula, Composita

Introduction

In this note we consider the Faber polynomials (see papers [1, 10, 2, 11]). Suppose we have the following generating function

$$\Phi(z) = \frac{1}{z} + \varphi_0 + \varphi_1 z + \ldots + \varphi_n z^n + \ldots$$

Then, raising this generating function to the power of k, we get

$$\Phi(z)^k = \Phi_k(z) + E_k(z),$$

where $\Phi_k(z)$ contains the coefficients with negative powers of z and $E_k(z)$ contains remaining coefficients. According to [10], $\Phi_k(z)$ is the Faber polynomial.

Main results

In this section, we present the main results of this paper.

Theorem 1. *Suppose we have a generating function* $g(t) = t + g_0 + \frac{g_1}{t} + \ldots + \frac{g_n}{t^n} + \ldots$. *Then the Faber polynomials of the generating function* $g(t)$ *are defined by*

$$F_n(x) = n \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} V^{\Delta}(n-k+j, j)\, x^{k-j}, \tag{1}$$

where $V^{\Delta}(n,k)$ *are coefficients of*

$$\left(t\, g\left(\frac{1}{t}\right) - 1\right)^k = \sum_{n \geqslant k} V^{\Delta}(n,k) t^n$$

Proof. Let us consider the generating function for Faber polynomials [7, 8]

$$\ln(1 + t\, v(t) - t\, x) = -\sum_{n>0} F_n(x)\, \frac{t^n}{n},$$

where $v(t) = t \cdot g\left(\frac{1}{t}\right) - 1$. To derive an explicit formula for the Faber polynomials we use the method for obtaining coefficients of generating function which is given by the composition of generating functions [3, 4]:
1. First, we find the following coefficients $A(t)^k = (t(v(t)-x))^k = \sum_{n\geqslant k} A^{\Delta}(n,k)t^n$:

$$(t(v(t)-x))^k = t^k \sum_{j=0}^{k} \binom{k}{j} v(t)^j (-1)^{k-j} x^{k-j}.$$

Then

$$A^{\Delta}(n,k) = \sum_{j=0}^{k} \binom{k}{j} V^{\Delta}(n-k+j,j)(-1)^{k-j} x^{k-j}.$$

2. According to the formula for composition of generating functions, we get the expression for the composition $b(x) = \ln(1+tv(t)-xt)$

$$b_n = \sum_{k=1}^{n} A^{\Delta}(n,k)\frac{(-1)^{k-1}}{k}.$$

Then the Faber polynomials are equal to

$$F_n(x) = -n\, b(n) = n \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} V^{\Delta}\left(n-k+j,j\right) x^{k-j}.$$

□

The obtained formula (1) allows us to give a method for obtaining the explicit expression and some interesting properties of the Faber polynomials.

Let us derive the explicit formula for $F_n(0)$. Using formula (1) with $k = j$, $x^{k-j} = 1$, we get

$$F_n(0) = \sum_{k=1}^{n} \frac{(-1)^k}{k} V^{\Delta}\left(n,k\right).$$

Now we give some examples.

Example 2. *Suppose we have the following generating function*

$$x + e^{\frac{1}{x}} = x + \sum_{n\geqslant 0} \frac{1}{n!x^n}.$$

Then $v(t) = t\, e^t$ *and*

$$V^{\Delta}(n,k) = \frac{k^{n-k}}{(n-k)!}.$$

Therefore, the Faber polynomials for the given generating function are defined by

$$F_n(x) = n \sum_{k=1}^{n} \frac{1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{j^{n-k}}{(n-k)!} x^{k-j}.$$

Example 3. *Suppose we have the following generating function*

$$\frac{3x - \sqrt{(x^2-4x)}}{2} = x + 1 + \frac{1}{x} + \frac{2}{x^2} + \frac{5}{x^3} + \dots$$

Then

$$v(t) = \frac{1 - \sqrt{1-4x}}{2}$$

and

$$V^{\Delta}(n,k) = \frac{k}{n}\binom{2\,n-k-1}{n-1}.$$

According to $V^{\Delta}(0,0) = 1$ *and after transformation, we get*

$$F_n(x) = \sum_{k=0}^{n} T(n,k)x^k,$$

where $T(n,k)$ *is the following triangle*

$$T(n,k) = \begin{cases} 0, & n = k = 0, \\ 1, & n = k, \\ \frac{n}{n-k} \sum\limits_{i=1}^{n-k} \frac{i\,(-1)^i \binom{k+i}{k}\binom{2\,(n-k)-i-1}{n-k-1}}{k+i}, & n > k. \end{cases}$$

Below we present first few elements of the triangle

$$\begin{array}{rrrrrr} 0 & & & & & \\ -1, & 1 & & & & \\ -1, & -2, & 1 & & & \\ -4, & 0, & -3, & 1 & & \\ -13, & -4, & 2, & -4, & 1 & \\ -46, & -10, & -5, & 5, & -5, & 1 \end{array}$$

Next we consider some integer properties of the Faber polynomials.

Corollary 4. *If the generating function* $g(t) = t + g_0 + \frac{g_1}{t} + \ldots + \frac{g_n}{t^n} + \ldots$ *has integer coefficients, then the Faber polynomials have integer coefficients.*

Using the results obtained in [9], we get the following property.

Corollary 5. *For the Faber polynomials of the generating function*

$$g(t) = t + g_0 + \frac{g_1}{t} + \ldots + \frac{g_n}{t^n} + \ldots$$

with integer coefficients the expression

$$\frac{F_n(m) - (g_0)^n}{n} \tag{2}$$

is integer for every prime n *and integer* m*.*

Conclusion

In this paper we study the Faber polynomials. We show a way to get explicit formulas for those polynomials. Also we present some properties and examples related to Faber polynomials.

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Some results for Bernstein type rational operators

Dilek Söylemez

In this talk, we focus on some generalizations of the Bleimann, Butzer, and Hahn operators. We investigate various properties of these operators, particularly, uniform convergence, and monotonicity properties of them. Furthermore, we discuss Korovkin type approximation results for these operators, employing specific summability methods, and determine the rate of the convergence using the modulus of continuity.

2020 MSC: 40A35, 40G10, 41A36

KEYWORDS: Korovkin type theorem, Bleimann, Butzer and Hahn operators, Summability methods

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Degenerate q-derangement numbers and polynomials

Dae Sik Lee

Some physicists or mathematicians have shown effective ways to find combinatorial numbers arising from normal ordering problems, which began in the work of Navon.

Let a and $a^\dagger$ be the Boson annihilation and creation operators satisfying the commutation relation:

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1.$$

Since $[a, a^\dagger] = [\frac{d}{dx}, x] = 1$, we use both forms by identifying formally $a = \frac{d}{dx}$ and $a^\dagger = x$.

When we consider the action of $(x\frac{d}{dx})^n$ on a polynomials $f(x)$, we have

$$\left(x\frac{d}{dx}\right)^n f(x) = \sum_{k=0}^{n} S_2(n,k)x^k \left(\frac{d}{dx}\right)^k f(x),$$

or, alternatively

$$(a^\dagger a)^n = \sum_{k=0}^{n} S_2(n,k)(a^\dagger)^k a^k.$$

Katriel showed that an integral power of the normal ordering $a^\dagger a$ can be written in the form

$$(a^\dagger a)^k = \sum_{l=0}^{k} S_2(k,l)(a^\dagger)^l a^l,$$

where $S_2(n,l)$ are the Stirling numbers of the second kind.

A derangement is a permutation with no fixed point. In other words, a derangement is a permutation of elements of a set that leaves no elements in their original places. The number of derangement of a set of size n is called the n-th derangement number and denoted by d_n.

The first few terms of the derangement number sequence $\{d_n\}_{n=0}^{\infty}$ are $d_0 = 1,\ d_1 = 0,\ d_2 = 1,\ d_3 = 2,\ d_4 = 9,\ d_5 = 44,\ d_6 = 265,\ d_7 = 1854,\ d_8 = 14833, \ldots$. In general, d_n is given by

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \quad (n \geq 0) \tag{1}$$

(*cf.* [9]).

The aim of this paper is to introduce degenerate q-derangement numbers and polynomials, which are the degenerate q-analogues of the derangement numbers and polynomials, and to investigate their connection with some other degenerate q-special numbers and polynomials.

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A note on the recent development of positive-linear operators involving special polynomials

Erkan Agyuz

The aim of this study is to bring together two areas in which positive linear operator theory and generating functions method. We will give a brief introduction to positive linear operators including generating functions of polynomials Bernoulli polynomials, Euler polynomials, and Fubini type polynomials. We will also discuss the convergence properties of these operators.

2020 MSC: 05A15, 47A58, 47B65

KEYWORDS: Generating functions, Positive linear operators, Euler polynomials, Fubini-type polynomials, Bernoulli polynomials

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$((p,q),\omega)$-Hahn difference operator and some important characterizations

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Postquantum calculus, also referred to as (p,q)-calculus, is an extension of quantum calculus that retrieves outcomes as p approaches 1. In this work, we aim to define a new operator namely, $((p,q),\omega)$-Hahn difference operator and establish a calculus on this operator by analyzing several structural features of this family, such as $((p,q),\omega)$-integral, $((p,q),\omega)$-derivative, product rule, quotient rule, linearity property, limit relations, and Leibniz rule.

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KEYWORDS: $((p,q),\omega)$-difference operator, $((p,q),\omega)$-integral, Leibniz rule

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Formulas derived from relationships between derangement numbers and the Peters-type Simsek numbers of the second kind

Elif Bozo [*1] *and Yilmaz Simsek* [2]

One of the purposes of this presentation is to study the relationships between the number of derangements and the Peters-type Simsek numbers of the second kind with their generating functions. The other purpose is to find certain family of finite sum, which is involving the n-th derangement number, which can be representation of $n!$ with the aid of the generating functions.

2020 MSC: 05A15, 11B75, 11B83, 60G50

KEYWORDS: Generating functions, Derangement numbers, Finite sums, Peters-type Simsek numbers and polynomials of the second kind

Introduction

Derangement numbers and polynomials

It is well-known that a derangement is not only a permutation, but also no fixed points. That is, a derangement (permutations without fixed points) is a permutation of the elements of a set that leaves no elements in their original places. The number of derangements of a set of size n is called the n-th derangement number. One of the well-known standard recurrences relation for the number of derangements are denoted by d_n on any set which has n (with $n \geq 1$) elements is given by

$$d_n = nd_{n-1} + (-1)^n \tag{1}$$

(*cf.* [2, p. 171], [9, p. 97]). By multiply both sides of Eq. (1) by $\frac{t^n}{n!}$ and sum over n to obtain

$$\sum_{n=0}^{\infty} d_n \frac{t^n}{n!} - 1 = t\sum_{n=1}^{\infty} d_{n-1}\frac{t^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!} - 1.$$

After some elementary calculation, the number of derangements are defined by means of the following generating function:

$$h(t) = \frac{1}{1-t}e^{-t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!} \tag{2}$$

(*cf.* [9, Exrcise 10, p. 391]).

The number of derangements have many applications in the topics of enumerative combinatorics analysis (*cf.* [2, p. 171], [8], [9, p. 97]).

By using (2), for $n \in \mathbb{N}_0$, one can easily obtain the following computation formula for the numbers d_n:

$$d_n = n!\sum_{j=0}^{n} \frac{(-1)^j}{j!} \tag{3}$$

(*cf.* [2, p. 171], [9, p. 97]). By using (3), few values of the numbers d_n are given by

$$d_0 = 1, d_1 = 0, d_2 = 1, d_3 = 2, d_4 = 9, \ldots$$

and so on.

Recently many researchers have investigated and studied on properties of this generating function. Because these numbers have many applications not only in combinatorics, which is the branch of mathematics that deals with different ways of selecting objects from a set or arranging objects, but also in other branches of mathematics, mathematical physics, and other applied sciences (*cf.* [2, p. 171], [3], [5], [8], [9, p. 97]).

Peters-type Simsek numbers and polynomials of the second kind

In [7], the generating function of Peters-type Simsek numbers and polinomials of the second kind $Y_{n,2}(x;\lambda)$, which is a member of the Peter polynomials family, which has recently spread rapidly in the field of generator functions:

$$F_{Y_2} = \frac{2(1+\lambda)^x}{\lambda^2 t + 2(\lambda-1)} = \sum_{n=0}^{\infty} Y_{n,2}(x;\lambda)\frac{t^n}{n!}. \tag{4}$$

When $x = 0$, Eq. (4) reduces to the Petes-type Simsek numbers of the second kind, which are given by means of the following generating function:

$$G_{Y_2}(t;\lambda) = \frac{2}{\lambda^2 t + 2(\lambda-1)} = \sum_{n=0}^{\infty} Y_{n,2}(x;\lambda)\frac{t^n}{n!} \tag{5}$$

such that

$$Y_{n,2}(\lambda) = Y_{n,2}(0;\lambda)$$

(*cf.* [7]).

For $n \in \mathbb{N}_0$, by combining (4) and (5), we have the following formula:

$$Y_{n,2}(x;\lambda) = \sum_{j=0}^{n} \binom{n}{j} x(x-1)(x-2)\cdots(x-j+1)Y_{n-j,2}(\lambda),$$

where the number $Y_{n,2}(\lambda)$

$$Y_{n,2}(\lambda) = 2(-1)^n \frac{\lambda^{2n} n!}{(2\lambda-2)^{n+1}} \tag{6}$$

and the polynomial $Y_{n,2}(x,\lambda)$ are given by

$$Y_{n,2}(x;\lambda) = 2\sum_{j=0}^{n} (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda-2)^{j+1}} (x)_{n-j} \tag{7}$$

(*cf.* [7, Lemma 1, p. 6 and Theorem 4, p. 6]).

Main results

In this section, by using definitions of the Peters-type Simsek numbers of the second kind, $Y_{n,2}(\lambda)$ and the number of derangements, d_n, we give some interesting results and formulas involving $n!$.

By using (5), we get

$$(1-\lambda)\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!} = -\sum_{n=0}^{\infty}\left(\frac{\lambda^2}{(2-2\lambda)}\right)^n \frac{d_n t^n}{n!}\sum_{n=0}^{\infty}\frac{\left(\lambda^2 t\right)^n}{(n!)\left(2(1-\lambda)\right)^n}.$$

By applying the Cauchy product to the above equation, we get

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n} d_k \frac{\lambda^{2n}}{2^n(1-\lambda)^n}\frac{t^n}{k!(n-k)!}\frac{n!}{n!} = (1-\lambda)\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

Therefore

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} d_k \frac{\lambda^{2n}}{2^n(1-\lambda)^{n+1}}\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the following theorem:

Theorem 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k} k! \sum_{j=0}^{k}\frac{(-1)^j}{j!} = n!. \tag{8}$$

so that

$$\sum_{k=0}^{n}\binom{n}{k} d_k = n!. \tag{9}$$

Using (8), we get

$$\sum_{k=0}^{n}\frac{1}{(n-k)!}\sum_{j=0}^{k}\frac{(-1)^j}{j!} = 1.$$

Combining the above equation with (3), we arrive at the following result:

Corollary 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\frac{d_k}{(n-k)!k!} = 1.$$

Conclusion

Our future plans are summuraize as follows:

Using (15) with (2), some functional equation and derivative equation will be investigate. Using these functions and equations, formulas for the Simsek-type numbers ($Y_{n,2}(\lambda)$) and the number of derangements (d_n) will also be examined. Moreover, using the definitions of the Simsek type numbers $Y_{n,2}(\lambda)$ and the derangement numbers d_n, we will examine another notation of $n!$ and the functional equations of the generating function, as well as many new formulas and relationships.

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I am very lucky to be your student. Thank you so much for all your support and everything so far. I dedicate this article to Professor SIMSEK's 60th birthday.

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Generalized Simsek sums and related certain family of finite sums

Elif Cetin

In [29] Simsek defined the function $\mathscr{F}(t,q)$. With the help of this function, he constructed generating functions for the q-Hardy-Berndt sums of the classical Hardy-Berndt sums $S(h,k), s_2(h,k), s_3(h,k)$, and $s_5(h,k)$. The objective of this study is to introduce the generalized Simsek sums, achieved by utilizing the function $\mathscr{F}(t,q)$ in conjunction with a new generating function. After the introduction of a generalization of the Simsek sums obtained by aid of q-analysis, then we present generalizations of some other special finite sums by using q-analysis methods.

2020 MSC: 11F20, 11C08, 05A15

KEYWORDS: Special finite sums, Generating functions, Hardy sums, Simsek sums, $C(h,k;1)$ sums, $B_1(h,k)$ sums

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On brief history and certain aspects of Cahn-Hilliard equations

Eylem Ozturk

The Cahn-Hilliard equation is a fundamental model that describes the phase separation process in multi-component mixtures. The Cahn-Hilliard equation was proposed in the late 1950s and has become central in understanding phase transition phenomena in many complex materials nowadays. It has been successfully extended to different contexts in various scientific fields. In this talk, we will briefly review the history, derivation, structure, and some applications of the Classical Cahn-Hilliard equation for motivation. Additionally, we will compare the classical Cahn-Hilliard equation with various variations of the equation that involve a fractional chemical potential. We will analyze the existence, uniqueness, and regularity of weak solutions, as well as the long-time behavior of weak solutions in the sense of global attractors.

2020 MSC: 45K05, 35B40, 37L30, 35B36

KEYWORDS: Cahn-Hilliard equation, Fractional Laplacian, Memory

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Some certain families of special polynomials derive from linear matrix form of transformation the using of rational numbers

Ezgi Polat [*1] *and Yilmaz Simsek* [2]

The goal of this presentation is to study and investigate new computational formulas of certain families of special polynomials derive from linear matrix form of the certain family of linear transformation on the ring of rational numbers. By using matrix operations, we give some examples for these polynomials. Moreover, we try to find general computational formulas in order to compute explicit values of these polynomials on ring of rational numbers.

2020 MSC: 05A15, 11B68, 47L05

KEYWORDS: Generating functions, Bernoulli polynomials and numbers, linear map

Introduction

In recent years, we have witnessed that Bernoulli polynomials and numbers, as well as their generating functions, are used effectively in almost many different scientific disciplines. Again, we know that Bernoulli polynomials and numbers are constructed using different methods or techniques. One of the most important techniques is to give the properties or explicit calculations formulas of these polynomials and numbers by using generating functions. It is well known that the construction of these generating functions can also be done using complex analysis methods, p-adic Volkenborn integral technique, and also other mathematical or physical techniques.

Unlike the above techniques, in this presentation we study and investigate inverse matrix representation of certain family of linear map on the set of rational numbers.

Throughout this presentation, our motivation is give not only the Bernoulli polynomials, but also their properties with aid of matrix representation of certain family of linear map on the set of rational numbers.

The $\mathbb{Q}$-linear map $E : \mathbb{Q}[\mathbf{x}] \to \mathbb{Q}[\mathbf{x}]$, from the polynomial ring $\mathbb{Q}[\mathbf{x}]$ to itself, is defined as follows:

$$E(P_n(x)) = \int_x^{x+1} P_n(y)dy, \tag{1}$$

where $P_n(y)$ denotes polynomial of degree n (*cf.* [1, p. 55]).

By using same method, which was given by [1, p. 55], the matrix representation of the $\mathbb{Q}$-linear map E with respect to the standard basis $\{1, x, x^2, \dots\}$ of $\mathbb{Q}[\mathbf{x}]$ is given by

$$E(x^n) = \frac{1}{(n+1)}\left((x+1)^{n+1} - x^{n+1}\right). \tag{2}$$

For $n = 3$, after some elementary calculations in the above equation, we have

$$E(x^3) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to give $B_0(x)$, $B_1(x)$, $B_2(x)$, and $B_3(x)$, we have

$$\left(\left(E(x^3)\right)^{-1}\right)^T \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} B_0(x) \\ B_1(x) \\ B_2(x) \\ B_3(x) \end{bmatrix}, \tag{3}$$

where $\left(\left(E(x^3)\right)^{-1}\right)^T$ denotes transpose of the matrix$\left(\left(E(x^3)\right)^{-1}\right)$ and $B_n(x)$ denotes the Bernoulli polynomials, which are defined by the following generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{4}$$

When $x = 0$, the above generating function reduces to the that of the Bernoulli numbers. Thus, we easily see that $B_n := B_n(0)$ (*cf.* [1]-[6]).

By using (4), we also have

$$\left(\left(E(x^3)\right)^{-1}\right)^T \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 \\ x - \frac{1}{2} \\ x^2 - \frac{3}{2}x + \frac{1}{6} \\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{bmatrix}.$$

By applying $D_x = \frac{d}{dx}$ derivative operator to (2), we get

$$D_x \left\{E(x^n)\right\} = \sum_{k=0}^{n-1} \binom{n}{k} x^k.$$

Consequently, we arrive at the following open problem:

Open problem: *Is it possible to give relation between* $D_x \left\{E(x^n)\right\}$ *and*

$$D_x \left\{B_n(x)\right\} = nB_{n-1}(x) \ ?$$

Conclusion

In the near future, our first aim is to investigate answer of this open problem. Our second aim is to study generalization of Equation (3). Moreover, using the same method of our paper, which was given in [4], we can plan to carry out detailed studies and investigations on new family of polynomials including the Bernoulli polynomials with the help of the inverse matrix representation of a certain family of linear map on the set of rational numbers.

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A note on generalized Szâsz-Schurer-Baskakov operator with their applications

Elif Sukruoglu *[1] *and Yilmaz Simsek* [2]

In this presentation is to study on the Stancu operator which is a generalization of the Bernstein operator which is also known as the Bernstein polynomials. We also give some remarks and comments on the generalized Szâsz-Schurer-Baskakov operator. We show that some special cases of these operators can be related to the Bernestein basis functions and the Stirling numbers.

2020 MSC: 33B10, 30G35, 41A10, 41A36

KEYWORDS: Bernstein polynomials, Szâsz-Schurer-Baskakov type basis function, Generating functions, Stancu operator

Introduction

Let f be a function on $[0,1]$. The Bernstein polynomials of degree n are defined by

$$B_m f(x) = \sum_{v=0}^{m} f\left(\frac{v}{m}\right) B_{v,m}(x); \quad 0 \leq x \leq 1,$$

where $B_m f(x)$ is called the Bernstein operator and

$$B_{v,m}(x) = \binom{m}{v} x^v (1-x)^{m-v}; \quad v = 0, 1, \ldots, n$$

are called the Bernstein basis polynomials (*cf.* [9]).

The Bernstein basis functions (*cf.* [7]):

$$B_k^n(x;a,b) = \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}$$

can be defined by the following moment generation function, ordinary generation function, and exponential generation function, respectively:

$$M_X(t,x;n;b,a) = \left(e^t \frac{x-a}{b-a} + \frac{b-x}{b-a}\right)^n,$$

$$\sum_{k=0}^{n} B_k^n(x) t^k = \left((1-x) + tx\right)^n; \quad a \leq x \leq b$$

and

$$\frac{t^k (x-a)^k}{(b-a)^n k!} e^{(b-x)t} = \sum_{n=0}^{\infty} B_k^n(x;a,b) \frac{t^n}{n!}$$

(*cf.* [6, 8]).

Schurer defined the Bernstein-Schurer operators for any $n \in \mathbb{N}$, $f \in C[0,1+p]$ and non-negative integers p using well-known the Bernstein operators. The Bernstein-Schurer operators are given by as follows:

$$B_{n,p}f(x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n}\right)\binom{n+p}{k} x^k (1-x)^{n+p-k}; \quad 0 \le x \le 1,$$

where $C[0,1+p] \to C[0,1]$ (*cf.* [5, 10, 11]).

The Baskakov-type base functions

$$y_v(a,b;m,n) = \frac{(-1)^v \binom{n+v-1}{v} \left(\frac{x-a}{b-x}\right)^v}{(b-x)^n (b-a)^m} \tag{1}$$

are defined by means of the following generating function:

$$A_t(x;a,b,m,n) = \sum_{v=0}^{\infty} y_v(a,b;m,n) t^v \tag{2}$$

(*cf.* [8]).

Substituting $m=n$, $a=0$, $b=1$ into (1), we get

$$y_v(x,0,1;n,n) = (-1)^v \binom{n+v-1}{v} \frac{x^v}{(1-x)^{n+v}}$$

and

$$y_v(-x,0,1;n,n) = \binom{n+v-1}{v} x^v (1-x)^{-n-v},$$

which gives us the Baskakov functions. Using (1), we have

$$(b-a)^n y_v(x;a,b;n,n) = (-1)^v \binom{n+v-1}{v} \left(\frac{x-a}{b-x}\right)^v.$$

The Catalan numbers are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{3}$$

where $n \ge 0$. The generating function for the Catalan numbers is given by

$$\frac{1-\sqrt{1-4t}}{n+1} = \sum_{n=0}^{\infty} C_n t^n,$$

where $0 < |t| < \frac{1}{4}$ and $C_0 = 1$ (*cf.* [2]).

The Catalan-type numbers are given by the following explicit formula:

$$V_n(\lambda) = (-1)^n C_n \frac{2^{n+1}\lambda^{2n}}{(\lambda-1)^{2n+1}} \tag{4}$$

which is defined by the following ordinary generating function:

$$\frac{1-\lambda+\sqrt{(\lambda-1)+8\lambda^2 t}}{2\lambda^2 t} = \sum_{n=0}^{\infty} V_n(\lambda) t^n$$

where $0 < \left|\frac{\lambda^2}{(\lambda-1)^2}\right| \le \frac{1}{2}$ (*cf.* [3]).

Main results

In this section, we give our main results.

The Baskakov-Schurer-Szâsz type linear positive operator is given by

$$L_{n,p}(f,x) = (n+p)\sum_{k=0}^{\infty} b_{n,p}^{k}(x)\int_{0}^{\infty} f(t)S_{n,p}^{k}(t)dt,$$

where $p, n, k \in \mathbb{N}$,

$$b_{n,p}^{k}(x) = \binom{n+p+k-1}{k}\frac{x^k}{(1+x)^{n+p+k}}$$

and

$$S_{n,p}^{k}(t) = \frac{((n+p)t)^k}{k!e^{-(n+p)t}}$$

(*cf.* [4, 1]). Thus

$$L_{n,p}(f,x) = \frac{1}{n+p}\mathcal{L}\left[f\right]\left(\frac{u}{n+p}\right) + (n+p)\sum_{k=1}^{\infty} b_{n,p}^{k}(x)\int_{0}^{\infty} f(t)S_{n,p}^{k}(t)dt, \qquad (5)$$

where $\mathcal{L}\left[f\right](t)$ denotes the Laplace transform of the function $f(t)$ for $t > 0$.

For special values of the function $f(t)$, we have the following results:

Substituting $f(t) = t^{\alpha-1}$ with $\alpha > 0$ into (5), we get

$$L_{n,p}(t^{\alpha-1},x) = (n+p)\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{x^k}{(1+x)^{n+p+k}}\frac{(n+p)^k}{k!}\int_{0}^{\infty} t^{\alpha-1+k}e^{(n+p)t}dt.$$

By applying the Mellin transformation to the above equation, after some calculations, we obtain

$$\begin{aligned} L_{n,p}(t^{\alpha-1},x) &= \frac{(n+p)}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k \\ &\times \frac{(n+p)^{k-1}}{k!}\frac{\Gamma(\alpha+k)}{(n+p)^{\alpha-1+k}}. \qquad (6)\end{aligned}$$

Substituting $\Gamma(\alpha+k) = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)\Gamma(\alpha)$ into (6), we get the following theorem:

Theorem 1. *Let* $\left|\frac{x}{1+x}\right| < 1$. *Then we have*

$$\begin{aligned} L_{n,p}(t^{\alpha-1},x) &= \frac{(n+p)^{1-\alpha}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k \\ &\times\frac{1}{k!}\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)\Gamma(\alpha). \qquad (7)\end{aligned}$$

Substituting $\alpha = \frac{1}{2}$ into (7), we get

$$L_{n,p}\left(t^{-\frac{1}{2}},x\right) = \frac{\sqrt{n+p}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k\Gamma\left(\frac{1}{2}+k\right)\frac{1}{k!}.$$

Using well known the Gamma function provides the equality,

$$\Gamma\left(\frac{1}{2}+k\right)=\frac{(2k)!}{(k!)^2}\frac{\sqrt{\pi}}{4^k},$$

Then we have

$$L_{n,p}\left(t^{-\frac{1}{2}},x\right)=\frac{\sqrt{n+p}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k\frac{1}{k!}\frac{(2k)!}{(k!)^2}\frac{\sqrt{\pi}}{4^k}. \tag{8}$$

Substituting $\alpha=n$, into (7), for $n\in\mathbb{N}$, we have

$$L_{n,p}(t^{n-1},x)=\frac{(n+p)^{1-n}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k\Gamma(n+k).$$

Thus, we get the following corollary:

Corollary 2. *Let $n,p\in\mathbb{N}$ and $\left|\frac{x}{1+x}\right|<1$. Then we have*

$$L_{n,p}(t^{\alpha-1},x)=\frac{(n+p)^{1-n}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k\frac{(n+k-1)!}{k!}. \tag{9}$$

Let classical derivative operator $D=\frac{d}{dx}$ and the Euler operator $Q=x\frac{d}{dx}$, we set

$$Q^*=(1+x)\frac{d}{dx},$$

where $(Q^*)^n=(Q^*)(Q^*)^{n-1}$ and

$$Q^*=D+Q.$$

Thus

$$(Q^*)^n=\left((1+x)^{n+p}\right)=(n+p)^n(1+x)^{n+p}.$$

Combining the operator to equation (9), we get the following corollary:

$$L_{n,p}(t^{n-1},x)=\frac{(n-1)!}{(Q^*)^n(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\binom{n+k-1}{n-1}\left(\frac{x}{1+x}\right)^k.$$

Using (3) and (8), we have the following formulas including the Catalan numbers:

Corollary 3. *Let $n,p\in\mathbb{N}$ and $\left|\frac{x}{1+x}\right|<1$. Then we have*

$$L_{n,p}\left(t^{-\frac{1}{2}},x\right)=\frac{\sqrt{n+p}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\left(\frac{x}{1+x}\right)^k\frac{(k+1)C_k}{k!4^k}.$$

Combining (4) with the Corollary 3, we also get the following result:

Corollary 4. *Let $n,p\in\mathbb{N}$, $\left|\frac{x}{1+x}\right|<1$ and $\lambda\neq1$. Then we have*

$$\begin{aligned}L_{n,p}\left(t^{-\frac{1}{2}},x\right)&=\frac{(\lambda-1)\sqrt{n+p}}{(1+x)^{n+p}}\sum_{k=0}^{\infty}(-1)^k\binom{n+p+k-1}{k}\left(\frac{x(\lambda-1)^2}{(1+x)\lambda^2}\right)^k\\&\quad\times\frac{(k+1)}{k!2^{3k+1}}V_k(\lambda).\end{aligned}$$

Acknowledgments

I would like to thank and extend my respects to my esteemed advisor, Professor Yilmaz SIMSEK, who has always helped, guided and enlightened me with his knowledge and experience in my academic life. I would like to dedicate this article to Professor Yilmaz SIMSEK on the occasion of his 60th birthday.

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Analyze of EEG and eye-tracking on tennis HEC training

Esra Suzen [*1], *Ovunc Polat* [2] *and Sukru Ozen* [3]

Ball from behind catch training is one of the commonly used exercises to improve eye-hand coordination in tennis. In this study, the success of catching during this training was evaluated based on EEG and eye tracking in 20 balls (Test-1) and 30 balls (Test-2) of 10 volunteers. The EEG sub-band energy values of the volunteers with low error rates decreased in the Test-2 exercise and those values increased in the high error volunteers. It was found that volunteers with low error rate minimized their movement on the x-axis and provided focus only in the y-direction, while volunteers with a high error created a sub-focus to exclude the eye-tracking and scanned a large area during the exercise.

2020 MSC: 42C40, 92C55

KEYWORDS: Biomedical signal processing, Electroencephalography, Gaze tracking, Wavelet transforms

Introduction

Hand-eye coordination (HCE) is very important in some special skills such as athletes and surgery. There are programs to develop visual-motor capabilities called sport vision training (SVT). A number of studies on sports have shown that visual-cognitive and visual-cognitive abilities have been developed in professional athletes [10, 11]. This coordination, which can be associated with the person's focus time and mental fatigue, is the basis for faster responses and more advanced motor responses.

Fleck et al. evaluated EEG-based changes in brain connectivity after exposure to bilateral eye movements. They stated that posterior delta coherence over time may be an important factor in maintaining attention [5]. An important part of HEC research is on eyes and saccadic gaze control, which is an important factor for motor control coordination [2, 6, 9].

The saccadic control is a critical parameter for HEC because both the eye and the hand movement fit produce a targeted optimized movement. [1] Optimal performance is achieved when eye movement occurs before motor movement and attempts to obtain the most visual clues about the task [7]. Visual cues have important knowledge about our cognitive and sensory motor behaviors. Visual scan patterns are associated with changes in the focus of attention.

HEC can be improved with SVT. Although SVTs contain different techniques, they include three main points: visual aspects, visual functions modified by training, and reflection of developments in visual abilities [4]. The most commonly used methods for HEC analysis are optical-based eye tracking mechanisms. Recently, studies on neural monitoring have increased [8].

In this study, the EEG and eye tracking changes that will occur in normal groups were analyzed by performing the catching exercise without seeing the ball which is one of the commonly used HEC trainings in tennis. Here, our main aim is to determine the electrophysiological changes in the brain created by repeated errors and increased periodic activity and to reveal its relationship with eye movements.

Methods

Ethic statement:

The study was approved by the Medical Ethical Committee of the Akdeniz University (715 - 06.12.2017) and the experiment was undertaken in compliance with national legislation and the Declaration of Helsinki. All the participants provided written, informed consent prior to the measurements.

Participants:

Ten adults (2 female and 8 male) with a mean of 33 ± 8.055 (mean ± standart deviation) years, ranging from 25 to 42 years, participated in the study. Participants were selected from people who did not exercise regularly, had no health problems and were right hand.

Firstly, volunteers were asked to catch the ball without touching the ground after bouncing off the wall. Tennis balls were randomly thrown into the black frame by a person behind the volunteer. The first training was performed with 20 balls (Test-1) and the second with 30 balls (Test-2). Between the two tests, 1 minute rest period was applied. It was marked whether he was able to catch the ball with each shot during the shots. Eye movements with Tobii Dynavox PCEye Mini eye tracker device and frontal cortex brain activities were recorded at a sampling frequency of 256 Hz with Muse EEG device. Data was recorded from four channels, four located on the scalp at locations AF7, AF8, TP9, TP10 as defined by the international 10-20 system and referenced to FPz.The EEG electrode locations are shown Fig. 1 and the test set-up is shown in Fig. 2.

EEG spectral analyzes were evaluated by decomposing at level 7 using db32 wavelet packet decomposition (WP).

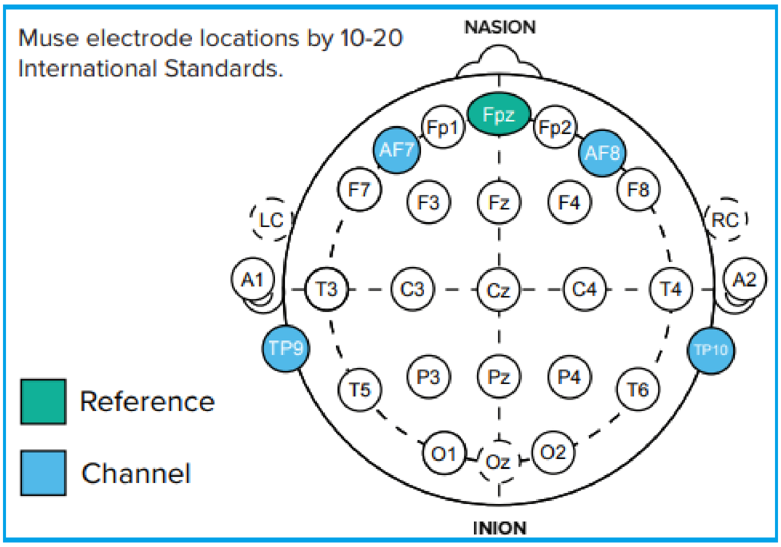


Figure 1: EEG electrode locations

Figure 2: Test set-up

WP is one of the most effective mathematical solution methods used in the analysis of non-stationary signals. In WP, both the approximation and the detail coefficients are decomposed at each level. WP is described as

$$W_{m,j,n}(t) = 2^{-m/2} W_j \left(2^{-m} t - n\right) \tag{1}$$

where $m, n \in \mathbb{Z}$ control wavelet dilation and translation respectively. $j \in \mathbb{N}$ denotes node index in each m level. [3]

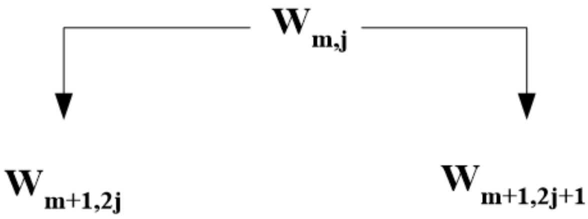


Figure 3: The decomposition for a node

The RMS energy in each node is determined by equation (2).

$$w_{rms,m,j}(t) = \sqrt{\frac{1}{N} \sum |w_{m,j}(r)|^2}. \tag{2}$$

Frequency characteristics of both the WP.

$$f_m = \frac{(j+1) f_s}{2^{m+1}}, \quad m = 1, \ldots, M-1 \tag{3}$$

where f_m is frequency in mth level, f_s is sampling frequency. Range of j is denoted as $j = 0, 1, \ldots, 2m - 1$ for WP. [3]

Wavelet packet nodes and frequency values for EEG bands for the 256 Hz sampling frequency are listed in Table 1.

EEG Bands	**Nodes**	**Frequency Range (Hz)**
Delta	w7,1 – w7,4	0 - 4
Teta	w7,5 – w7,8	4 - 8
Alfa	w7,9 – w7,13	8 - 13
Beta	w7,14 – w7,30	13 - 30
Gama	w7,31 – w7,50	30 - 50

Table 1: The decomposition for a node

Results

The balls that cannot be held by the volunteers in Test-1 and Test -2 are shown as % error value in Fig. 4.

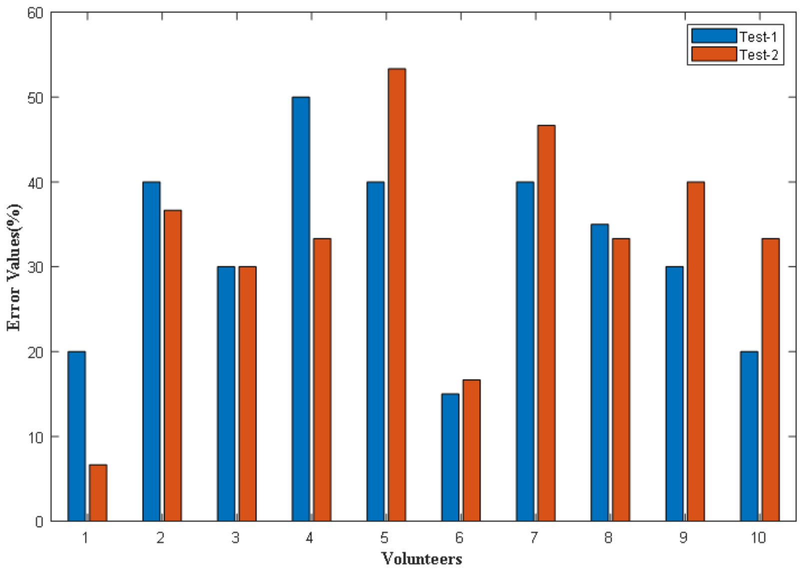


Figure 4: Error values (%) for Test-1 and Test-2

The EEG sub-band energy values obtained for Test-1 and Test -2 were compared and the results were evaluated as the increase-reduction value. (Fig. 5.) A similar characteristic of 90% and above was determined when the energy characteristics of the 5 bands in the EEG were analyzed by this increase-reduction rate. Test-2 energy values of delta, theta, beta and gamma bands were decreased in the AF7 electrode of 5 subjects whose mean error was below 35%. In 4 out of 5 people with an error above 35%, EEG energy values in these bands were found to be higher than test 1 in test 2.

In the alpha band, in 4 out of 5 people with a mean error of less than 35%, test-2 AF7 energy values are higher than test-1. In the same band, 4 of the 5 people with an average error of more than 35% had increased AF7 energy values in test-2.

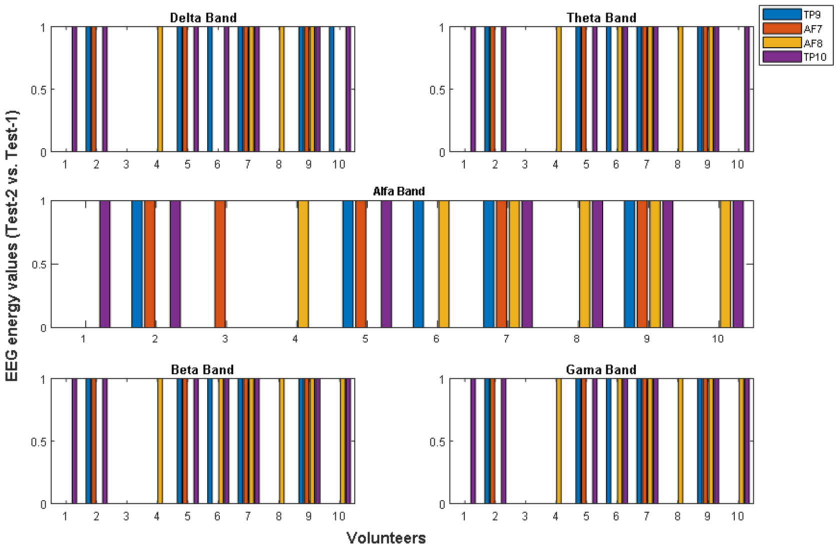
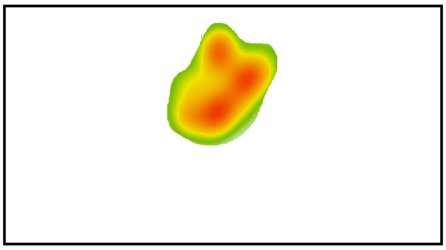
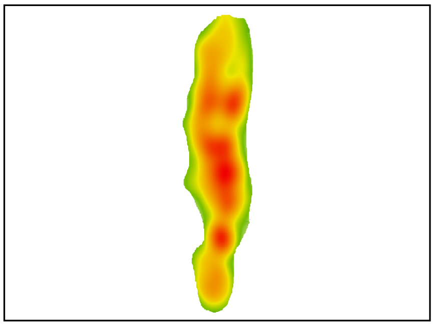


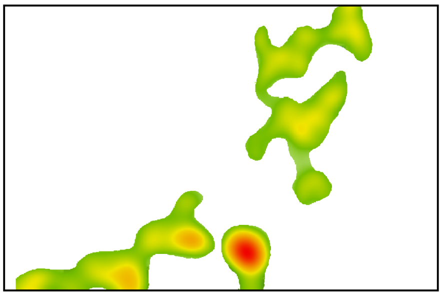
Figure 5: EEG energy values for each sub-bands. If test-2 energy values are higher than test-1 energy values, it is marked as 1 in the graph and 0 if it is smaller



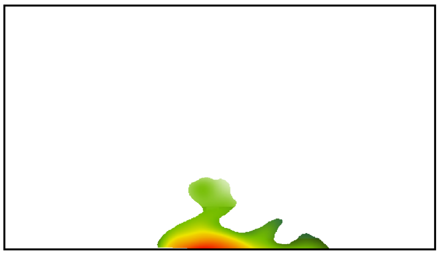
a) Volunteer-1, error rate: 6.67%, Test-2



b) Volunteer-6, Error rate: 16.67%-Test-2



c) Volunteer-7, error rate: 40%-Test-1



d) Volunteer-5, error rate: %53.33, Test-2

Figure 6: As an example, the eye tracking heat maps of 2 people with high error and 2 persons with low error

Test-2 energy values Compared to Test-1, 7 of 10 volunteers increased the TP10 Test-2 energy values. No significant difference was found in AF8 and TP9.

In the eye tracking heat maps, it was found that the two people who made the least mistakes, scanning the limited area along the y axis and restricting their movement in the x-axis. It was determined that 3 people with error above 45% have moved out of the boundaries of the eye tracking device, creating a focus at a lower point and moving their eyes in the whole frame. The results of the eye-tracking heat map are for two high and two low-voluntary volunteers were shown in Fig. 6.

Conclusion

Eye-hand coordination is very important in order to increase success in sports activities and to be a professional athlete. Therefore, training techniques are used to develop eye-hand coordination for each sport. This necessitates the follow-up of the athlete during the training sessions, the analysis of the characteristics of the high-performing people, and the planning of new training programs and durations within the other athletes. In this study, the ball catching exercise was designed in 2 different ways. Our main objective is to evaluate the focus ability of the person in the increasing number of balls with eye tracking and to detect the electrophysiological change in the brain with EEG.

The error values (%) of 4 volunteers participating in the study decreased in Test-2. This value increased in 5 volunteers. Only 2 people with a percentage of error below 20%. The average error of Test-1 is 32 ± 11.105 and Test-2 is 33 ± 13.465. In sport, the athlete is expected to focus on viewing the whole area at the same time. It was determined that the volunteers who made an error of less than 35% performed this process with minimized movement on the x-axis. These people continued their movement on the y-axis. It was seen that the highest percentage of the error groups carried the focus centers to a lower point outside the area and continuously scanned the whole area in the x-y axis. It was considered that this could create a focusing problem during the game.

In addition, the EEG activities in all sub-bands of volunteers with high error rates in Test-2 increased compared to Test-1.

This study, which is carried out both as electrophysiological and as eye tracking for movement and focus tracking, will be performed with data such as accelerometer and HRV on larger groups and will contribute to the creation of the right training programs.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Regression analysis of heavy metal content of Macroalgae in the Gulf of Antalya

Emine Sukran Okudan

Investigation of heavy metal contents in macroalgae and evaluation of their statistical data have gained importance in recent years. In this study, it was aimed to make regression analyses and interpretation of heavy metal contents of Macroalgae of Antalya Gulf. SPSS 22 statistical software package was used for the interpretation of heavy metal contents. According to regression analysis, Durbin-Watson= 2.269, $R^2 = 0.999$ and Sig.= 0 were calculated. Regression analysis was useful in determining the adequacy of the data and the accuracy of the analysis results. In this context, it is recommended to apply regression analysis in similar scientific studies.

2020 MSC: 62J05, 92B10, 62M10

KEYWORDS: Macroalgae, Regression analysis, Heavy metal, Antalya Gulf

Introduction

In the study and evaluation of natural environments, heavy metal researches of natural, anthropogenic and industrial origin are quite high. Especially the researches on water, soil, rock and mining sites are quite high [4, 9, 10, 13]. Similar studies with macroalgae have started to gain importance as much as these studies [1, 2, 3, 5, 6, 7, 8, 11, 12].

There are many studies on different subjects related to the Gulf of Antalya, which constitutes the study area. However, there are not enough studies on macroalgae. In this context, the Gulf of Antalya was chosen as the study area. The aim of this study is to perform and interpret regression analysis from multivariate statistical analyses on the heavy metal contents of different and/or same species of macroalgae in the shelf area of the Gulf of Antalya.

Materials and methods

Chemical analyses of macroalgae samples from different locations of the Gulf of Antalya were carried out using X-Ray fluorescence (XRF) method. Regression analysis was applied to the chemical analysis data obtained from multivariate statistical analyses.

Results and discussion

Regression analysis was applied to the chemical analysis results of macroalgae samples taken from the sea. When the "Model Summary[b]" from the regression analysis was examined, the R Square (R^2) value was calculated as 0.999. This value showed that the data obtained from the study area were sufficient. "Durbin-Watson" value was calculated as 2.269. The range required for the model is between −3 and +3 and it is important that the value is around +2 (Table 1).

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	0.999[a]	0.999	0.991	2778.2	2.269

a. Predictors: (Constant), C, V, Cu, S, Zn, In, Te, Hf, P, Yb, Ta, Ga, U, Mg, Cr, I, Mn, Cl, As, Pa, Ni, Au, Ba, Dy, Pb, Sn, Co, Al, K, Br, Sr, Tm, Er, Se, Re, Y, Rb, Na, Tb, Fe, Eu, Zr, Ti, Ca

b. Dependent Variable: Si

Table 1: Model Summary[b]

When "ANOVAa" was analysed from the regression analysis, the "the significance level" value was 0.000. We can say that there is no error value for the data and they are reliable data (Table 2).

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	48018760065.058	44	1091335456.024	141.395	0.000[b]
	Residual	69465284.196	9	7718364.911		
	Total	48088225349.253	53			

a. Dependent Variable: Si

b. Predictors: (Constant), C, V, Cu, S, Zn, In, Te, Hf, P, Yb, Ta, Ga, U, Mg, Cr, I, Mn, Cl, As, Pa, Ni, Au, Ba, Dy, Pb, Sn, Co, Al, K, Br, Sr, Tm, Er, Se, Re, Y, Rb, Na, Tb, Fe, Eu, Zr, Ti, Ca

Table 2: ANOVAa

When Residuals Statistic[a] is analysed, "Std. Residual" is between −1.109 and +1.155. It is between −2 and +2. The significance for the data is high. However, "Cook's Distance" is between 0 and 9.369. In order to avoid outliers, the value of 9.369 should be expected to be zero. Although not very high, outliers are considered to be effective in this analysis (Table 3).

	Minimum	Maximum	Mean	Std. Deviation	N
Predicted Value	1109,7505	132068,4844	29152,9556	30100,07211	54
Std. Predicted Value	-0,932	3,419	0,000	1,000	54
Standard Error of Predicted Value	1841,386	2776,131	2525,775	231,127	54
Adjusted Predicted Value	-3561,9749	110215,2656	30126,9078	29396,71762	54
Residual	-3082,39648	3209,57568	0,00000	1144,84311	54
Std. Residual	-1,109	1,155	0,000	0,412	54
Stud. Residual	-2,963	2,440	-0,048	1,083	54
Deleted Residual	-57087,44141	54934,96484	-973,95227	15023,34677	54
Stud. Deleted Residual	-17,746	3,953	-0,309	2,711	54
Mahal. Distance	22,302	51,940	43,185	7,595	54
Cook's Distance	0,000	9,369	0,615	1,763	54
Centered Leverage Value	0,421	0,980	0,815	0,143	54

a. Dependent Variable: Si

Table 3: Residuals statisticsa

In the linear model, parameter estimation was performed. It was found that Si value showed an increase of 0.13 to Fe content among the analysed samples. It is understood that the data move in accordance with the line and the distribution is symmetrical (Figure 1).

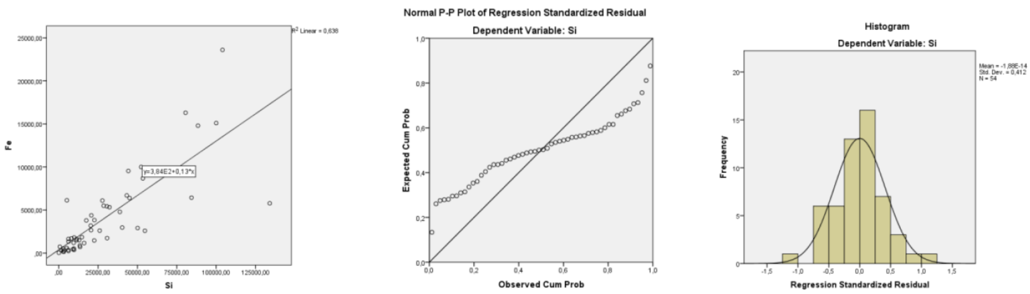

Figure 1: Distribution of chemical analysis results

Results

As a result of multivariate statistical analysis (Regression), Durbin-Watson= 2.269, $R^2 = 0.999$ and Sig.= 0.000 values were obtained statistically significant results by regression analysis in order to better explain the accuracy of the statistics made in the macroalgae samples of the Gulf of Antalya.

However, for every 1 ppm Si concentration increase, Fe element was found to increase and a model was proposed; $R^2 = 0.638$ and $d_{Fe} = -3,84 + 0,13d_{Si}$.

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Wasserstein bounds in the CLT of the MLE for the drift coefficient of a stochastic partial differential equation

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In this work, we are interested in the rate of convergence for the central limit theorem of the maximum likelihood estimator of the drift coefficient for a stochastic partial differential equation based on continuous time observations of the Fourier coefficients $u_i(t),\ i = 1,\ldots,N$ of the solution, over some finite interval of time $[0,T]$. We provide explicit upper bounds for the Wasserstein distance for the rate of convergence when $N \to \infty$ and/or $T \to \infty$. In the case when T is fixed and $N \to \infty$, the upper bounds obtained in our results are more efficient than those of the Kolmogorov distance given by Mishra and Prakasa Rao and Kim and Park.

2020 MSC: 62F12, 60F05, 60G15, 60H15, 60H07

Keywords: Parameter estimation, Stochastic partial differential equations, Rate of normal convergence of the MLE, Wasserstein distance

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Ruled surfaces in spacetime case characterized by a stationary disteli-axis

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The aim of this paper is to obtain the equations that describe spacelike ruled surfaces having a stationary Disteli-axis, using the E. Study map. These equations allow us to calculate a group of Lorentzian curvature functions that determine the shape of these surfaces in their local region. As a result, we can derive and analyze some familiar surface theory formulas in Lorentzian line space and their geometric interpretations. Finally, the paper develops and explores a detailed description of the properties that define a spacelike line trajectory as a constant Disteli-axis.

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KEYWORDS: Disteli-axis, Serret-Frenet motion, Striction curve

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On Solé and Planat criterion for the Riemann hypothesis

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There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{q \leq q_n} (1 + \frac{1}{q}) > e^{\gamma} \cdot \log \theta(q_n)$ holds for all prime numbers $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and log is the natural logarithm. In this note, using Solé and Planat criterion, we prove that the Riemann hypothesis is true.

2020 MSC: 11M26, 11A41, 11A25

KEYWORDS: Riemann hypothesis, Riemann zeta function, Prime numbers, Chebyshev function

Introduction

The Riemann hypothesis is the assertion that all non-trivial zeros have real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm.

Proposition 1. *We have [6, pp. 1]:*

$$x \sim \theta(x) \quad when \quad (x \to \infty).$$

Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 2. *It is known that [1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number).

In mathematics, $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

Next, we have Solé and Planat Theorem:

Proposition 3. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann hypothesis is true [7, Theorem 4.2 pp. 5].*

A natural number N_k is called a primorial number of order k precisely when,

$$N_k = \prod_{i=1}^{k} q_i.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ is satisfied. There are several statements out from the Riemann hypothesis assumption:

Proposition 4. *We have [7, Proposition 3. pp. 3]:*

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

On the sum of the reciprocals of all prime numbers not exceeding x, we have

Proposition 5. *For* $x \geq 2278383$ *[2, Theorem 5.6 (1) pp. 243]:*

$$-\frac{0.2}{\log^3 x} \leq \sum_{q \leq x} \frac{1}{q} - B - \log \log x \leq \frac{0.2}{\log^3 x}$$

where $B \approx 0.2614972128$ *is the Meissel-Mertens constant [4, (17.) pp. 54].*

The following property is based on natural logarithms:

Proposition 6. *[3, pp. 1]. For* $x > 0$*:*

$$\log(1 + x) < x.$$

Putting all together yields a proof for the Riemann hypothesis using the Chebyshev function.

What if the Riemann hypothesis were false?

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Lemma 7. *If the Riemann hypothesis is false, then there are infinitely many prime numbers* q_n *for which* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. The Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [7, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [5, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [5, Theorem 3 (c) pp. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

Main insight

Theorem 8. *The Riemann hypothesis is true when for every large enough prime number $q_n > 3$, there exists another prime $q_{n'} > q_n$ such that*

$$R(N_{n'}) \leq R(N_n).$$

Proof. If the Riemann hypothesis is false and the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for every large enough prime number $q_n > 3$, then there is an infinite subsequence of natural numbers n_i such that

$$R(N_{n_{i+1}}) \leq R(N_{n_i}),$$

$q_{n_{i+1}} > q_{n_i}$ and $\mathsf{Dedekind}(q_{n_i})$ fails by Lemma 7. This is a contradiction with the fact that

$$\liminf_{k\to\infty} R(N_k) = \lim_{k\to\infty} R(N_k) = \frac{e^\gamma}{\zeta(2)}$$

by Proposition 4. By definition of the limit inferior for any positive real number ε, only a finite number of elements of the sequence $R(N_k)$ are less than $\frac{e^\gamma}{\zeta(2)} - \varepsilon$. This is a contradiction with the previous infinite subsequence and thus, the Riemann hypothesis must be true. □

Main theorem

Theorem 9. *The Riemann hypothesis is true.*

Proof. The Riemann hypothesis is true when

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for large enough prime numbers $q_{n'} > q_n$ because of the Theorem 8. That is the same as

$$\log\log\theta(q_{n'}) - \log\log\theta(q_n) \geq \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right).$$

By Propositions 1 and 5, for every large enough prime number $q_n > 3$, there exists another prime $q_{n'} > q_n$ such that

$$\begin{aligned}
&\log\log\theta(q_{n'}) - \log\log\theta(q_n)\\
&\approx \log\log(q_{n'}) - \log\log(q_n)\\
&= \frac{0.2}{\log^3 q_n} + B + \log\log(q_{n'}) - B - \frac{0.2}{\log^3 q_n} - \log\log(q_n)\\
&\gtrapprox \sum_{q_n < q \leq q_{n'}} \frac{1}{q}\\
&> \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right)
\end{aligned}$$

where

$$q_n \sim \theta(q_n) \quad when \quad (n \to \infty)$$

and the inequality $\frac{1}{q} > \log\left(1 + \frac{1}{q}\right)$ is satisfied for every prime q by Proposition 6. Consequently, the inequality

$$R(N_{n'}) \leq R(N_n)$$

holds for sufficiently large prime numbers $q_{n'} > q_n$ and therefore, the Riemann hypothesis is true. □

Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. In general, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics.

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NP on logarithmic space

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P versus NP is considered as one of the most important open problems in computer science. This consists in knowing the answer of the following question: Is P equal to NP? It was essentially mentioned in 1955 from a letter written by John Nash to the United States National Security Agency. However, a precise statement of the P versus NP problem was introduced independently by Stephen Cook and Leonid Levin. Since that date, all efforts to find a proof for this problem have failed. Another major complexity classes are L and NL. Whether $L = NL$ is another fundamental question that it is as important as it is unresolved. We prove that $NP \subseteq NSPACE(\log^2 n)$ just using logarithmic space reductions.

2020 MSC: 68Q15, 68Q17

KEYWORDS: Computational algorithm, Complexity classes, Completeness, Polynomial time, Reduction, Logarithmic space

Introduction

In 1936, Turing developed his theoretical computational model [8]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation [8]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [8]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [8].

Let Σ be a finite alphabet with at least two elements, and let Σ^* be the set of finite strings over Σ [1]. A Turing machine M has an associated input alphabet Σ [1]. For each string w in Σ^* there is a computation associated with M on input w [1]. We say that M accepts w if this computation terminates in the accepting state, that is $M(w) =$"*yes*" [1]. Note that, M fails to accept w either if this computation ends in the rejecting state, that is $M(w) =$"*no*", or if the computation fails to terminate, or the computation ends in the halting state with some output, that is $M(w) = y$ (when M outputs the string y on the input w) [1].

P is the complexity class of languages that can be decided by deterministic Turing machines in polynomial time [2]. A verifier for a language L_1 is a deterministic Turing machine M, where:

$$L_1 = \{w : M(w, u) = \text{“yes"} \textit{ for some string } u\}.$$

We measure the time of a verifier only in terms of the length of w, so a polynomial time verifier runs in polynomial time in the length of w [1]. A verifier uses additional information, represented by the string u, to verify that a string w is a member of L_1. This information is called certificate. NP is the complexity class of languages defined by polynomial time verifiers [6].

A function $f : \Sigma^* \to \Sigma^*$ is a logarithmic space computable function if some deterministic Turing machine M, on every input w, halts using logarithmic space in its work tapes with just $f(w)$ on its output tape [8]. Let $\{0,1\}^*$ be the infinite set of binary strings, we say that a language $L_1 \subseteq \{0,1\}^*$ is logarithmic space reducible to a language $L_2 \subseteq \{0,1\}^*$, written $L_1 \leq_l L_2$, if there is a logarithmic space computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for all $x \in \{0,1\}^*$:

$$x \in L_1 \textit{ if and only if } f(x) \in L_2.$$

An important complexity class is *NP–complete* [3]. If L_1 is a language such that $L' \leq_l L_1$ for some $L' \in$ *NP–complete*, then L_1 is *NP–hard* [2]. Moreover, if $L_1 \in NP$, then $L_1 \in$ *NP–complete* [2]. The *NP–complete* class is formally defined by polynomial time reductions [3]. A principal *NP–complete* problem is SAT [3].

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [8]. The work tapes may contain at most $O(\log n)$ symbols [8]. In computational complexity theory, L is the complexity class containing those decision problems that can be decided by a deterministic logarithmic space Turing machine [6]. NL is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [6].

In general, $DSPACE(S(n))$ and $NSPACE(S(n))$ are complexity classes that are used to measure the amount of space used by a Turing machine to decide a language, where $S(n)$ is a space-constructible function that maps the input size n to a non-negative integer [5]. The complexity class $DSPACE(S(n))$ is the set of languages that can be decided by a deterministic Turing machine that uses $O(S(n))$ space [5]. The complexity class $NSPACE(S(n))$ is the set of languages that can be decided by a nondeterministic Turing machine that uses $O(S(n))$ space [5].

We state the following Hypothesis:

Hypothesis 1. *There is an NP–complete language* $L_1 \in NSPACE(\log^2 n)$ *which is closed under logarithmic space reductions in NP–complete.*

We show the principal consequence of this Hypothesis:

Theorem 2. *If the Hypothesis 1 is true, then* $NP \subseteq NSPACE(\log^2 n)$.

Proof. Due to L_1 is closed under logarithmic space reductions in *NP–complete*, then every NP problem is logarithmic space reduced to L_1. This implies that $NP \subseteq NSPACE(\log^2 n)$ since $NSPACE(\log^2 n)$ is closed under logarithmic space reductions as well. □

The problems

Now, we define the problems that we are going to use.

Definition 3. ***SUBSET PRODUCT (SP)***

INSTANCE: A list of natural numbers B *and a positive integer* N*.*

QUESTION: Is there collection contained in B *such that the product of all its elements is equal to* N*?*

REMARKS: We assume that every element of the list divides N*. Besides, the prime factorization of every element in* B *and* N *is given as an additional data.* $SP \in$ *NP–complete [3].*

Definition 4. ***Unary 0–1 Knapsack (UK)***

INSTANCE: A positive integer 0^y *and a sequence* $0^{y_1}, 0^{y_1}, \ldots, 0^{y_n}$ *of positive integers represented in unary.*

QUESTION: Is there a sequence of 0–1 valued variables $x_1, x_2, \ldots, x_n$ such that

$$y = \sum_{i=1}^{n} x_i \cdot y_i?$$

REMARKS: We assume that the positive integer zero is represented by the fixed symbol 0^0. $UK \in NL$ [4].

Results

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Theorem 5. $SP \in NSPACE(\log^2 n)$.

Proof. Given an instance (B, N) of SP, then for every prime factor p of N we could create the instance

$$0^y, 0^{y_1}, 0^{y_1}, \ldots, 0^{y_n}$$

for UK such that $B = [B_1, B_2, \ldots, B_n]$ is a list of n natural numbers and $\nu_p(N) = y, \nu_p(B_1) = y_1, \nu_p(B_2) = y_2, \ldots, \nu_p(B_n) = y_n$ (Do not confuse n with N). Under the assumption that N has k prime factors, then we can output in logarithmic space each instance for UK such that these instances of UK appears in ascending order according to the ascending natural sort of the respective k prime factors. That means that the first UK instance in the output corresponds to the smallest prime factor of N and the last UK instance in the output would be defined by the greatest prime factor of N. Besides, in this logarithmic space reduction we respect the order of the exponents according to the appearances of the n elements of $B = [B_1, B_2, \ldots, B_n]$ from left to right: i.e. every instance is written to the output tape as

$$0^z, 0^{z_1}, 0^{z_1}, \ldots, 0^{z_n}$$

where $\nu_q(N) = z, \nu_q(B_1) = z_1, \nu_q(B_2) = z_2, \ldots, \nu_q(B_n) = z_n$ for every prime factor q of N. Finally, we generate a certificate on the work tapes that is a sequence of *0–1* valued variables $x_1, x_2, \ldots, x_n$ using **square logarithmic** space such that for the first instance of UK we have

$$y = \sum_{i=1}^{n} x_i \cdot y_i,$$

for the second one

$$z = \sum_{i=1}^{n} x_i \cdot z_i,$$

and so on...

We can simulate a composition in logarithmic space reduction that simultaneously accept the k instances of UK. We can do this since the sequence certificate would be exactly the same for the k instances of UK. Every logarithmic space computation uses $O(\log \mid (B, N) \mid)$ space where $\mid \ldots \mid$ is the bit length function. So, we finally consume $O(k \cdot \log \mid (B, N) \mid)$ space exactly in the whole computation that would be **square logarithmic** space because of $k \sim \log \log N$ and thus, the whole computation can be made $O(\log^2 \mid (B, N) \mid)$ space. If N is a primorial then $k \sim \frac{\log N}{\log \log N}$, but in that case the bit length of (B, N) is exponential over k when (B, N) is an acceptance instance of SP.

When we read one *0–1* valued variable x_i that is equal to 1 in the first instance of UK, then we store the current sum that includes adding the unary length of the element in the position i inside of the list. Next, we do the same for the remaining $k-1$ instances of UK for the elements in the same position i. We store each current sum only once over the work tapes in the contiguous k instances of UK while we simultaneously copy these instances to the output tape from left to right. After that, we place the input head again in the first instance of UK and check whether the next *0–1* valued variable x_{i+1} is equal to 1 or not on the work tapes (We do not do nothing if the current *0–1* valued variable is equal to 0). We repeat over and over again this process without moving the output tape to the left during this composition of logarithmic space reduction [6]. In fact, we copy to the output tape the consecutive k instances of UK during this composition of logarithmic space reduction exactly the same number of times that the *0–1* valued variables in the certificate are equal to 1. Note that, the output tape of the inner Turing machine is the input tape of the outer Turing machine during this composition in logarithmic space reduction.

To sum up, we can create this composition in logarithmic space reduction that only uses a **square logarithmic** space in the work tapes such that the sequence of variables is placed on the work tapes due to we can read at once every valued variable x_i and remove it later. Hence, we only need to iterate from the variables of the sequence from the indexes 1 to n to verify whether we generate an appropriate certificate according to the described constraints of the problem UK to finally accept the k instances otherwise we can reject.

In addition, we can simulate the reading of one symbol from the string sequence of *0–1* valued variables into the work tapes just nondeterministically generating the symbol in the work tapes using a **square logarithmic** space [1]. We could remove each symbol or a **square logarithmic** amount of symbols generated in the work tapes, when we try to generate the next symbol contiguous to the right on the string sequence of *0–1* valued variables. We could generate the certificate from the inner Turing machine in the composition of logarithmic space reduction and so, the outer Turing machine would be one-way deterministic during this composition of computations. In this way, the generation will always be in **square logarithmic** space. This proves that SP is in $NSPACE(\log^2 n)$. □

Theorem 6. *$NP \subseteq NSPACE(\log^2 n)$.*

Proof. This is a directed consequence of Theorems 2 and 5 because of the Hypothesis 1 is true. Certainly, SP is closed under logarithmic space reductions in *NP–complete*. Indeed, we can reduced SAT to SP in logarithmic space and every NP problem could be logarithmic space reduced to SAT by the Cook's Theorem Algorithm [3]. □

Conclusions

Savitch's theorem states that for any space-constructible function $S(n) \geq \log n$, we obtain that $NSPACE(S(n)) \subseteq DSPACE(S(n)^2)$ and therefore, $NSPACE(\log^2 n) \subseteq DSPACE(\log^4 n)$ [7]. Since $DSPACE(S(n))$ can be solved by a deterministic Turing machine in $O(2^{O(S(n))})$ time for any space-constructible function $S(n) \geq \log n$, then this would mean that $NP \subseteq QP$ (quasi-polynomial time class). We "believe" there must exist an evident proof of $NSPACE(\log^2 n) \subseteq P$ and thus, we would obtain that $P = NP$.

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Recent trends to the problem of heteroscedastity in regression Analysis

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One of the assumptions of the classical regression model, which is frequently used in statistical applications in social sciences, is that the error terms are homoscedasticity. In cases where this assumption is not met, there are some regulatory measures in the literature. The purpose of this article is to give a summary for regulatory measures in case of heteroscedasticity.

2020 MSC: 62-06, 62J05 , 62P25

KEYWORDS: Statistics, Regression Analysis, Heteroscedasticity

Introduction

One of the basic assumptions of both simple and multiple regression analysis is the constant variance assumption. This assumption means that the variance of the error terms of each observation is the same. If this assumption is not valid, the problem encountered is "heteroscedasticity" is named. In this case, the variance of each error is different. Having heteroscedasticity will affect parameter estimators in the regression model. Parameter estimators will not be "biased" and "effective". Estimating the parameter estimators different from the actual values will affect the interval estimates, t and F tests [1].

Material and method

The classical regression model is as follows equation (1)

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + u_i \\ i &= 1, 2, ..., n. \end{aligned} \tag{1}$$

The error term u in this model has the following characteristics equation (2)

$$\begin{aligned} E(u_i) &= 0 \\ var(u_i) &= \sigma^2 \\ cov(u_i, u_k) &= 0, \quad i \neq k. \end{aligned} \tag{2}$$

In the case of heteroscedasticity, the error term is as follows equation (3)

$$var(u_i) = \sigma_i^2 \tag{3}$$

that is, each error term has different variance (*cf.* [1, 2, 3], Figure1).

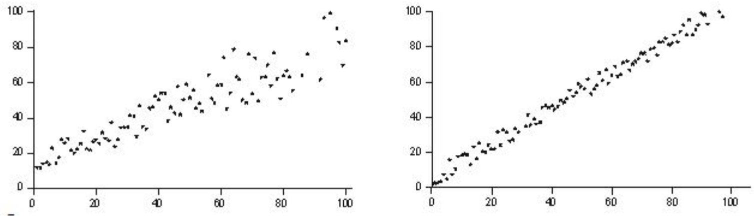

Figure 1: The first graph shows heteroscedasticity and the other graph shows homoscedasticity

Detecting Heteroscedasticity in Data

How to understand the presence of heteroscedasticity in a certain situation? Although there is no definite rule for this, there are formal and informal testing methods in the literature (*cf.* [1, 2, 3]). Some of these testing methods are as follows:

1. Informal methods
   - Nature of the problem
   - Drawing method
2. Formal methods
   - Park's test [4]
   - Glejser's test [5]
   - Spearman rank correlation test
   - Goldfeld-Quandt tests [6, 7]
   - Breusch-Pagan-Godfrey test [8, 9]
   - White's Test for heteroscedasticity [10].

Which test is best is an open question. Each test is based on different assumptions. William Green may be examined for details on this subject [11, 12].

Corrective measures for heteroscedasticity problem

If σ_i^2 is known, the shortest way to correct heteroscedasticity problem is the weighted least squares method, which is often used. It is a method of transforming the data by assigning different weights to the observed values. Let's take the simple linear model equation (4)

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + u_i \\ i &= 1, 2, ..., n. \end{aligned} \tag{4}$$

Since the variance is assumed to be known, the solution method given in equation (5) can eliminate heteroscedasticity problem.

$$\begin{aligned} \frac{Y_i}{\sigma_i} &= \beta_0^* \left(\frac{1}{\sigma_i}\right) + \beta_1^* \left(\frac{X_i}{\sigma_i}\right) + \frac{u_i}{\sigma_i} \\ var\left(\frac{u_i}{\sigma_i}\right) &= \frac{\sigma_i^2}{\sigma_i^2} = 1. \end{aligned} \tag{5}$$

In cases where σ_i^2 is not known, corrective methods are used depending on the shape of the existing problem. In general, there are 4 approaches in the literature.

Case 1. If the explanatory variable used in the model is proportional to the error variance, the suggested transformation is as in (6).

$$\frac{Y_i}{X_i} = \beta_0^* \left(\frac{1}{X_i}\right) + \beta_1^* \left(\frac{X_i}{X_i}\right) + \frac{u_i}{X_i}. \tag{6}$$

The prediction values obtained as a result of the application of this model will be the prediction values of the transformed model.

Case 2. If the error variance is directly proportional to the explanatory variable in the model, then the (7) transform is used.

$$\frac{Y_i}{\sqrt{X_i}} = \beta_0^* \left(\frac{1}{\sqrt{X_i}}\right) + \beta_1^* \left(\frac{X_i}{\sqrt{X_i}}\right) + \frac{u_i}{\sqrt{X_i}}. \tag{7}$$

Case 3. If the square of the mean values of Y is proportional to the error variance, the transformation to be made is the transformation in (8) .

$$\frac{Y_i}{E\sqrt{Y_i}} = \beta_0^* \left(\frac{1}{E\sqrt{Y_i}}\right) + \beta_1^* \left(\frac{X_i}{E\sqrt{Y_i}}\right) + \frac{u_i}{E\sqrt{Y_i}}. \tag{8}$$

Case 4. If the heteroscedasticity problem exists, logarithmic transformation of the data can be a solution.

$$\log Y_i = \beta_0 + \beta_1 \log X_i + u_i. \tag{9}$$

This transformation given with the equation (9) may be the first and the shortest solution that comes to mind, as well as the approach that eliminates the heteroscedasticity problem for many data sets.

Of course, the transformations given here for these approaches are for summary only. Researchers are recommended to read their articles for problems [13]. At the same time, Yalcin and Mert's article can be examined for an application example [14].

Conclusion

In this study, information scanned from the literature on detecting and correcting the heteroscedasticity problem was compiled. The solution methods for this problem continue to be developed by valuable researchers.

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A collocation method based on Lerch polynomials for solving the Hantavirus infection model

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In this paper, a collocation method for solving the Hantavirus infection model is presented. The basis function used for the collocation method are based on Lerch polynomials. According to method, the Lerch polynomials solutions of the Hantavirus infection model are written in matrix forms. By using the collocation points and matrix forms of required functions such as the derivatives of the approximate solutions, the nonlinear term and the initial conditions in the Hantavirus infection model, the method transforms the Hantavirus infection model into a system of nonlinear algebraic equations. The solutions of this nonlinear system determine the coefficients of assumed solution forms. In addition, by giving the error estimation method, the accuracy of the approximate solutions is tested. A numerical application for the model is made and the obtained results are compared with the available methods in literature. The application results are presented in tables and graphs. All applications are made using a code written in MATLAB.

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Keywords: Collocation method, Error estimation, Hantavirus infection model, Lerch polynomials, Nonlinear differential equations

Introduction

Hantavirus, an infectious diseases of animal origin, was firstly reported in Canada in 1994. Recently, many researchers have studied on the numerical solutions of the Hantavirus infection model such as the variational iteration method [11], the differential transformation method [12], an exponential matrix method [25]. A model of the infection of Hantavirus in deer mouse [1], two new mathematical models for hantavirus infection in rodents [3], a hyperbolic reaction–diffusion model for describing the hantavirus infection in mice population [7] have been studied by many researchers. Traveling waves in a model of the Hantavirus infection in deer mice have been analyzed [2].

The system of differential equations have been solved by using various numerical method such as an exponential matrix method [26], Laguerre collocation method [27, 28], Taylor collocation and Adomian decomposition method [5], rational Chebyshev collocation method [20], exponential Fourier collocation method [30], block hybrid collocation method [22] and etc. [21, 13, 10, 29].

On the other hand, many equations such as singular Fredholm integral equations [14], 2D and 3D Volterra type integral and second order partial integro differential equations [9], pure Neumann problem [8] have been numerically solved by using Lerch polynomials. These studies have been yielded effective results. But, the Lerch collocation method has not been studied for the Hantavirus infection model. In this paper, we develop a collocation method based on Lerch polynomials numerically to solve Hantavirus infection model given by

$$\begin{cases} S'(t) = x(S+I) - yS - \frac{S(S+I)}{z} - pSI, & \\ & 0 \le t < b < \infty \\ I'(t) = -yI - \frac{I(S+I)}{z} + pSI & \\ S(0) = d, \quad I(0) = f & \end{cases} \tag{1}$$

where x, y, z, p, d, f are the constant parameters.

$S(t)$ and $I(t)$ represent, respectively, the populations of the susceptible mice and the populations of the infected mice. d and f represent, respectively, the initial number of the susceptible mice and the initial number of the infected mice. z show the environmental parameter. x, y and p denote, respectively, infection rates, birth rates and death rates.

The approximate solutions of (1) are investigated as

$$\begin{aligned} S_N(t) &= \sum_{j=0}^{N} a_{1,j} L_j(t,\gamma) \\ I_N(t) &= \sum_{j=0}^{N} a_{2,j} L_j(t,\gamma) \end{aligned} \tag{2}$$

where $a_{1,j}$ and $a_{2,j}$ represent, respectively, the unknown coefficients of $S_N(t)$ and the unknown coefficients of $I_N(t)$. $L_j(t,\gamma)$ $(j = 0, 1, ..., N)$ are the Lerch polynomials defined by [8]

$$L_j(t,\gamma) = \sum_{i=1}^{j} \frac{i!}{j!} s(j,i) \binom{i+\gamma-1}{i} t^i. \tag{3}$$

Here, $L_0(t,\gamma) = 1$ is initial value, γ is a parameter and $s(j,i)$ is Stirling numbers of the first kind [6].

In this study, we also present an error estimation method. For this, we use the residual function and the Lerch polynomial solutions. Accordingly, we can comment on the error of the presented method.

Lerch collocation method

Firstly, we write the solutions (2) in matrix form:

$$\begin{aligned} S_N(t) &= \mathbf{L}(t,\gamma)\mathbf{A}_1 \\ I_N(t) &= \mathbf{L}(t,\gamma)\mathbf{A}_2 \end{aligned} \tag{4}$$

where $\mathbf{L}(t,\gamma) = \begin{bmatrix} L_0(t,\gamma) & L_1(t,\gamma) & \cdots & L_N(t,\gamma) \end{bmatrix}$,

$$\mathbf{A}_1 = \begin{bmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,N} \end{bmatrix}^T \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{2,0} & a_{2,1} & \cdots & a_{2,N} \end{bmatrix}^T.$$

Secondly, we express the Lerch polynomials in matrix form [8]

$$\mathbf{L}(t,\gamma) = \mathbf{T}(t)\mathbf{D}(\gamma) \tag{5}$$

where $\mathbf{L}(t,\gamma) = \begin{bmatrix} L_0(t,\gamma) & L_1(t,\gamma) & \cdots & L_N(t,\gamma) \end{bmatrix}$, $\mathbf{T}(t) = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix}$ and

$$\mathbf{D}(\gamma) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1!}{1!}s(1,1)\binom{\gamma}{1} & \frac{1!}{2!}s(2,1)\binom{\gamma}{1} & \cdots & \frac{1!}{N!}s(N,1)\binom{\gamma}{1} \\ 0 & 0 & \frac{2!}{2!}s(2,2)\binom{\gamma+1}{2} & \cdots & \frac{2!}{N!}s(N,2)\binom{\gamma+1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{N!}{N!}s(N,N)\binom{\gamma+N-1}{N} \end{bmatrix}.$$

By substituting the matrix relation (5) in (4), we write the Lerch polynomial solutions of (1) as follows:

$$\begin{aligned} S_N(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1 \\ I_N(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2. \end{aligned} \tag{6}$$

As a third, we take the derivative of (6) and so we obtain

$$\begin{aligned} S_N'(t) &= \mathbf{T}(t)\mathbf{B}^T\mathbf{D}(\gamma)\mathbf{A}_1 \\ I_N'(t) &= \mathbf{T}(t)\mathbf{B}^T\mathbf{D}(\gamma)\mathbf{A}_2 \end{aligned} \tag{7}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N & 0 \end{bmatrix}.$$

Using (6), we write nonlinear terms in the Hantavirus infection model (1) in the matrix forms:

$$\begin{aligned} S_N(t)S_N(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1, \\ S_N(t)I_N(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2, \\ I_N(t)S_N(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1, \\ I_N(t)I_N(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2. \end{aligned} \tag{8}$$

Fourthly, we express the matrix relations of the initial conditions $S(0) = d$ and $I(0) = f$ with the help of (6) as follows:

$$\begin{aligned} S_N(0) = d &\rightarrow \mathbf{T}(0)\mathbf{D}(\gamma)\mathbf{A}_1 = d \\ I_N(0) = f &\rightarrow \mathbf{T}(0)\mathbf{D}(\gamma)\mathbf{A}_2 = f \end{aligned} \tag{9}$$

or

$$\begin{aligned} \mathbf{U}\mathbf{A}_1 = d, &\quad \mathbf{U} = \mathbf{T}(0)\mathbf{D}(\gamma), \\ \mathbf{V}\mathbf{A}_2 = f, &\quad \mathbf{V} = \mathbf{T}(0)\mathbf{D}(\gamma). \end{aligned} \tag{10}$$

As the next step, we substitute the matrix relations (6), (7) and (8) in the Hantavirus infection model (1). Thus, we obtain the matrix relation

$$\begin{aligned} &\mathbf{T}(t)\mathbf{B}^T\mathbf{D}(\gamma)\mathbf{A}_1 - (x-y)\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1 - x\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2 \\ &\quad + \tfrac{1}{z}\left(\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\right) + \left(\tfrac{1}{z}+p\right)\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2 = 0, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\mathbf{T}(t)\mathbf{B}^T\mathbf{D}(\gamma)\mathbf{A}_2 + y\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2 + \tfrac{1}{z}\left(\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\right) \\ &\quad - \tfrac{1}{z}\left(\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2\right) + p\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2 = 0. \end{aligned} \tag{12}$$

Next, we define the collocation points as

$$t_i = \frac{b}{N} i, \quad i = 0, 1, ..., N. \tag{13}$$

By writing the collocation points t_i instead of t in (11)-(12), we get

$$\begin{bmatrix} \mathbf{W}_0\mathbf{A}_1 + \mathbf{C}_0\mathbf{A}_2 \\ \mathbf{K}_0\mathbf{A}_1 + \mathbf{H}_0\mathbf{A}_2 \\ \vdots \\ \mathbf{W}_N\mathbf{A}_1 + \mathbf{C}_N\mathbf{A}_2 \\ \mathbf{K}_N\mathbf{A}_1 + \mathbf{H}_N\mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

where

$$\begin{aligned}
\mathbf{W}_i &= \mathbf{T}(t_i)\mathbf{B}^T\mathbf{D}(\gamma) - (x-y)\mathbf{T}(t_i)\mathbf{D}(\gamma) + \tfrac{1}{z}\left(\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t_i)\mathbf{D}(\gamma)\right), \\
\mathbf{C}_i &= -x\mathbf{T}(t_i)\mathbf{D}(\gamma) + \left(\tfrac{1}{z} + p\right)\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t_i)\mathbf{D}(\gamma), \\
\mathbf{K}_i &= \tfrac{1}{z}\left(\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t_i)\mathbf{D}(\gamma)\right), \\
\mathbf{H}_i &= \mathbf{T}(t_i)\mathbf{B}^T\mathbf{D}(\gamma) + y\mathbf{T}(t_i)\mathbf{D}(\gamma) - \tfrac{1}{z}\left(\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t_i)\mathbf{D}(\gamma)\right) \\
&\quad + p\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t_i)\mathbf{D}(\gamma).
\end{aligned}$$

Finally, we write systems (10) and (14) as a single system. By solving the obtained new system in MATLAB, we have the Lerch coefficient matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. Then, we substitute the calculated coefficient matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ in (6) and we obtain the Lerch polynomial solutions of (1).

Error estimation method

In this section, firstly we substitute the Lerch polynomial solutions in (1) and from here we get the residual functions

$$\begin{cases} R_{1,N} = S_N' - x(S_N + I_N) + yS_N + \frac{S_N(S_N+I_N)}{z} + pS_N I_N, \\ R_{2,N} = I_N' + yI_N + \frac{I_N(S_N+I_N)}{z} - pS_N I_N, \\ S_N(0) = d, \quad I_N(0) = f. \end{cases} \tag{15}$$

Here, $S_N(t)$ and $I_N(t)$ represent the Lerch polynomial solutions of (1). $R_{1,N}(t)$ and $R_{2,N}(t)$ represent the residual functions of (1).

Secondly, we define the error functions for the Hantavirus infection model (1) as

$$\begin{aligned} e_{1,N}(t) &= S(t) - S_N(t), \\ e_{2,N}(t) &= I(t) - I_N(t). \end{aligned} \tag{16}$$

Here, $S(t)$ and $I(t)$ represent the exact solutions of (1). $e_{1,N}(t)$ and $e_{2,N}(t)$ represent the actual error functions of (1).

As a next step, we subtract (15) from Hantavirus infection model (1) and so we get the error problem in the form:

$$\begin{cases} e_{1,N}' = \left(x - y - \frac{2S_N+I_N}{z} - pI_N\right)e_{1,N} - \frac{1}{z}e_{1,N}^2 + \left(x - \frac{S_N}{z} - pS_N\right)e_{2,N} \\ \qquad + \left(-\frac{1}{z} - p\right)e_{1,N}e_{2,N}, \\ e_{2,N}' = \left(-y - \frac{2I_N+S_N}{z} + pS_N\right)e_{2,N} - \frac{1}{z}e_{2,N}^2 + \left(-\frac{I_N}{z} + pI_N\right)e_{1,N} \\ \qquad + \left(-\frac{1}{z} + p\right)e_{1,N}e_{2,N}, \\ e_{1,N}(0) = 0, \quad e_{2,N}(0) = 0. \end{cases} \tag{17}$$

This error problem is obtained by writing $e_{1,N}(t) + S_N(t)$ and $e_{2,N}(t) + I_N(t)$ instead of $S(t)$ and $I(t)$, respectively, with the help of (16).

Finally, we solve this error problem (17) according to the Lerch collocation method in Section 2. Hence, we obtain the estimated error functions

$$\begin{cases} e_{1,N,M}(t) = \sum_{i=0}^{M} a_{1,i}^{*} L_i(t,\gamma) \\ e_{2,N,M}(t) = \sum_{i=0}^{M} a_{2,i}^{*} L_i(t,\gamma). \end{cases} \tag{18}$$

Numerical results

In this section, we select the parameters of the Hantavirus infection model (1) as $x = 1, y = 0.5, z = 20, p = 0.1, d = 10, f = 10$. According to the selected parameters, we show the numerical results in tables and graphics.

Accordingly, the Hantavirus infection model (1) becomes

$$\begin{cases} S^{'}(t) = (S+I) - 0.5S - \frac{S(S+I)}{20} - 0.1SI, \\ I^{'}(t) = -0.5I - \frac{I(S+I)}{20} + 0.1SI, \\ S(0) = 10, \quad I(0) = 10. \end{cases} \tag{19}$$

According to the method in Section 2, we investigate the Lerch polynomial solutions for $N = 3$ as

$$\begin{aligned} S_3(t) &= \sum_{i=0}^{3} a_{1,i} L_i(t,\gamma) \\ I_3(t) &= \sum_{i=0}^{3} a_{2,i} L_i(t,\gamma) \end{aligned} \tag{20}$$

or

$$\begin{aligned} S_3(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_1 \\ I_3(t) &= \mathbf{T}(t)\mathbf{D}(\gamma)\mathbf{A}_2. \end{aligned} \tag{21}$$

Now, we determine the collocation points for the range $[0, 1]$ as $t_0 = 0, t_1 = 1/3, t_2 = 2/3, t_3 = 1$. From (14), we can write

$$\begin{bmatrix} \mathbf{W}_0\mathbf{A}_1 + \mathbf{C}_0\mathbf{A}_2 \\ \mathbf{K}_0\mathbf{A}_1 + \mathbf{H}_0\mathbf{A}_2 \\ \mathbf{W}_1\mathbf{A}_1 + \mathbf{C}_1\mathbf{A}_2 \\ \mathbf{K}_1\mathbf{A}_1 + \mathbf{H}_1\mathbf{A}_2 \\ \mathbf{W}_2\mathbf{A}_1 + \mathbf{C}_2\mathbf{A}_2 \\ \mathbf{K}_2\mathbf{A}_1 + \mathbf{H}_2\mathbf{A}_2 \\ \mathbf{W}_3\mathbf{A}_1 + \mathbf{C}_3\mathbf{A}_2 \\ \mathbf{K}_3\mathbf{A}_1 + \mathbf{H}_3\mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} &\mathbf{W}_i = \mathbf{T}(t_i)\mathbf{B}^T\mathbf{D}(\gamma) - 0.5\mathbf{T}(t_i)\mathbf{D}(\gamma) + \tfrac{1}{20}\left(\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t_i)\mathbf{D}(\gamma)\right), \\ &\mathbf{C}_i = -\mathbf{T}(t_i)\mathbf{D}(\gamma) + \left(\tfrac{1}{20} + 0.1\right)\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t_i)\mathbf{D}(\gamma), \\ &\mathbf{K}_i = \tfrac{1}{20}\left(\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t_i)\mathbf{D}(\gamma)\right), \\ &\mathbf{H}_i = \mathbf{T}(t_i)\mathbf{B}^T\mathbf{D}(\gamma) + 0.5\mathbf{T}(t_i)\mathbf{D}(\gamma) - \tfrac{1}{20}\left(\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_2\mathbf{T}(t_i)\mathbf{D}(\gamma)\right) \\ &\qquad +0.1\mathbf{T}(t_i)\mathbf{D}(\gamma)\mathbf{A}_1\mathbf{T}(t_i)\mathbf{D}(\gamma), \\ &\mathbf{A}_1 = \begin{bmatrix} a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \end{bmatrix}^T, \\ &\mathbf{A}_2 = \begin{bmatrix} a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}^T. \end{aligned}$$

On the other hand, the matrix relations of the initial conditions are

$$\begin{cases} \mathbf{T}(0)\mathbf{D}(\gamma)\mathbf{A}_1 = 10 \\ \mathbf{T}(0)\mathbf{D}(\gamma)\mathbf{A}_2 = 10. \end{cases} \tag{23}$$

Finally, we solve the obtained system by combining (22)-(23) with the help of MATLAB for $\gamma = 10$. Thus, we calculate the coefficient matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ and we write this coefficient matrices in (21). Consequently, we obtain the Lerch polynomial solutions.

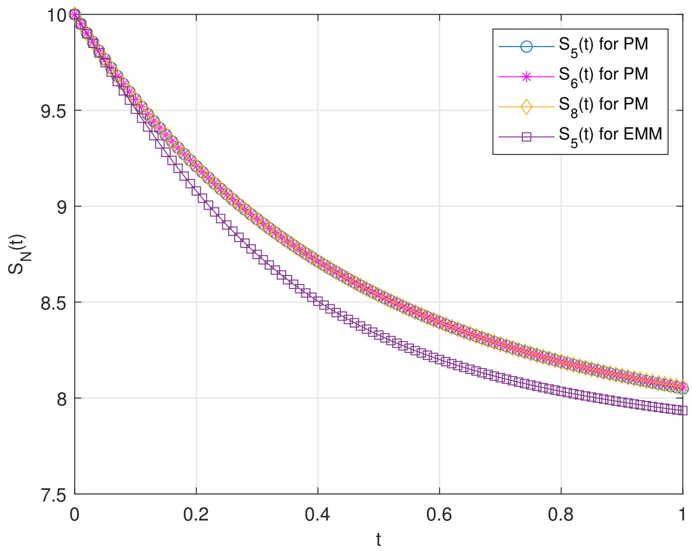

Figure 1: Plot of approximate solution $S_N(t)$ for the presented method and the exponential matrix method (EMM) [25]

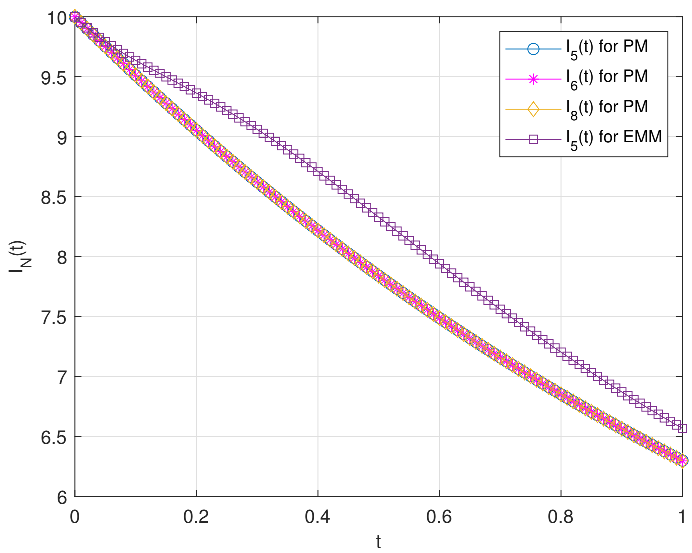

Figure 2: Plot of approximate solution $I_N(t)$ for the presented method and the exponential matrix method (EMM) [25]

Figure 1 compares the graphs of populations of the susceptible mice $S_N(t)$ for $N = 5$, $N = 6$, $N = 8$ with the exponential matrix method (EMM) [25] for $N = 5$.

Figure 2 compares the graphs of populations of the infected mice $I_N(t)$ for $N = 5$, $N = 6$, $N = 8$ with exponential matrix method (EMM) [25] for $N = 5$.

According to Figure 1 and Figure 2, it can be said that there is a decrease in the populations of susceptible mice and infected mice at 1 days.

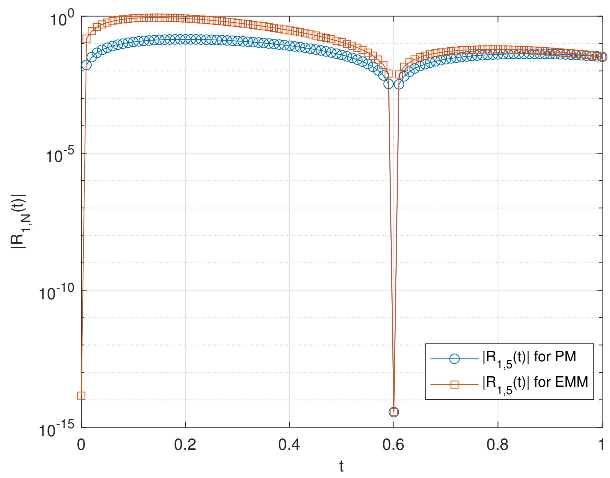

Figure 3: Plot of the absolute residual errors $|R_{1,5}(t)|$ for the presented method and the exponential matrix method (EMM) [25]

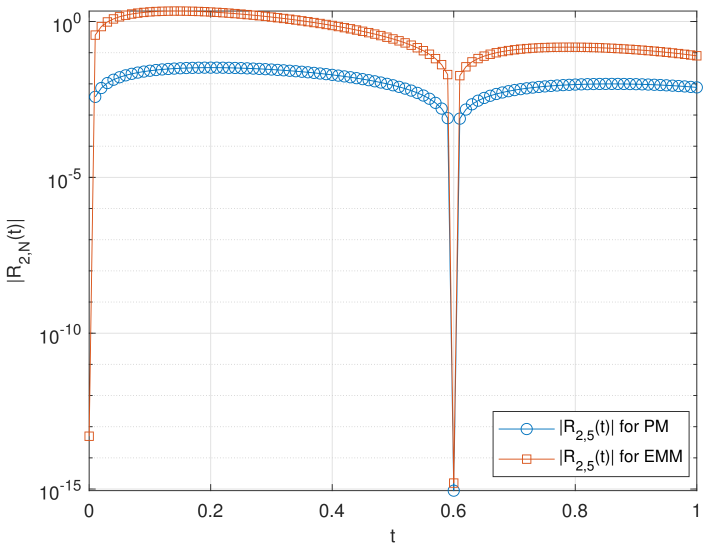

Figure 4: Plot of the absolute residual errors $|R_{2,5}(t)|$ for the presented method and the exponential matrix method (EMM) [25]

t_i	$\lvert R_{1,5}(t)\rvert$ for PM	$\lvert R_{2,5}(t)\rvert$ for PM	$\lvert R_{1,5}(t)\rvert$ for EMM	$\lvert R_{2,5}(t)\rvert$ for EMM
0	0	0	1.1990e-14	5.8842e-14
0.2	1.4223e-01	3.3179e-02	8.1824e-01	2.0207e+00
0.4	8.2809e-02	1.9460e-02	3.0114e-01	7.4703e-01
0.6	2.2649e-15	1.7645e-15	2.6645e-15	5.1070e-15
0.8	4.0311e-02	9.4979e-03	6.1037e-02	1.5027e-01
1	3.2816e-02	7.7090e-03	3.2497e-02	7.9540e-02

Table 1: Comparison of the absolute residual error with the exponential matrix method (EMM) [25]

Figure 3 compares the absolute residual errors of the solution $S_N(t)$ with the exponential matrix method (EMM) [25] for $N = 5$. Similarly, Figure 4 compares the absolute residual errors of the solution $I_N(t)$ with the exponential matrix method (EMM) [25] for $N = 5$. Also, Table 1 presents these absolute residual errors for various points of t_i.

It is said from Figure 3, Figure 4 and Table 1 that we obtain better results than the exponential matrix method (EMM) [25].

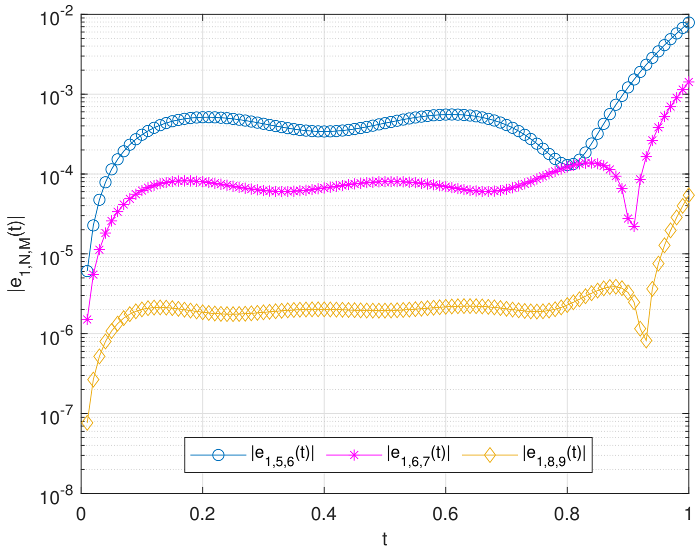

Figure 5: Plot of the absolute estimated errors $|e_{1,N,M}(t)|$

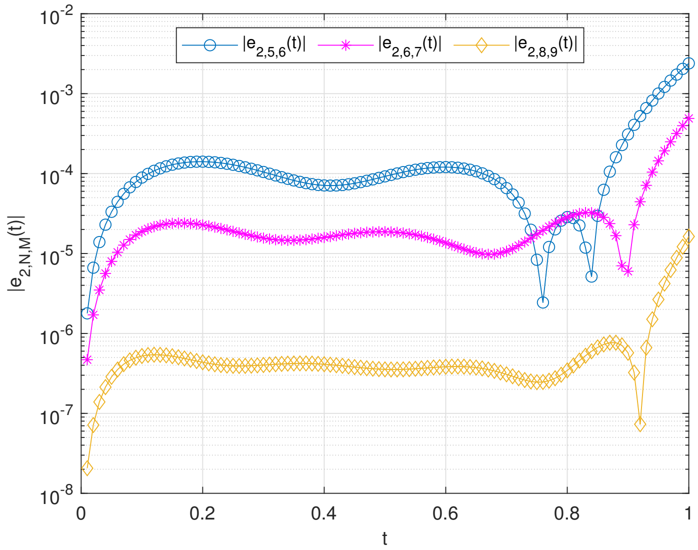

Figure 6: Plot of the absolute estimated errors $|e_{2,N,M}(t)|$

t_i	$\lvert e_{1,5,6}(t)\rvert$	$\lvert e_{1,6,7}(t)\rvert$	$\lvert e_{1,8,9}(t)\rvert$	$\lvert e_{2,5,6}(t)\rvert$	$\lvert e_{2,6,7}(t)\rvert$	$\lvert e_{2,8,9}(t)\rvert$
0	0	0	0	0	0	0
0.2	5.1400e-04	8.0010e-05	1.8594e-06	1.4100e-04	2.2790e-05	4.3605e-07
0.4	3.4079e-04	6.6868e-05	2.0217e-06	7.1118e-05	1.5878e-05	4.1204e-07
0.6	5.5499e-04	6.9357e-05	2.1738e-06	1.2055e-04	1.3567e-05	3.8319e-07
0.8	1.2985e-04	1.2255e-04	2.2889e-06	2.8593e-05	2.8413e-05	3.4493e-07
1	7.8853e-03	1.4212e-03	5.4267e-05	2.3927e-03	4.9000e-04	1.6337e-05

Table 2: Comparison of the absolute estimated error for $(N, M) = (5, 6)$, $(N, M) = (6, 7)$ and $(N, M) = (8, 9)$

Figure 5 and Figure 6 show, respectively, the absolute estimated errors of the solution $S_N(t)$ and the absolute estimated errors of the solution $I_N(t)$ for $(N, M) = (5, 6), (N, M) = (6, 7)$ and $(N, M) = (8, 9)$. Similarly, Table 2 presents these absolute estimated errors for various points of t_i.

It is interpreted from Figure 5, Figure 6 and Table 2 that the errors decrease while the value of N in the method increases. Hence, the results of the absolute estimated errors are quite satisfactory. This error estimation method is important. Because we can comment on the error made in the method if we do not know the exact solution of the model.

Conclusion

In this paper, we present the Lerch collocation method for the approximate solutions of the Hantavirus infection model. Also, we develop an error estimation method by using the residual function. By selecting the parameters of the Hantavirus infection model as $x = 1, y = 0.5, z = 20, p = 0.1, d = 10, f = 10$, we apply the Lerch collocation method to the Hantavirus infection model with the help of a code written in MATLAB. According to the selected parameters, we give the obtained numerical results in table and graphs. Accordingly, the errors decrease as the value of N in the method increases. The error estimation method is quite successful. In addition, we make a comparison with exponential matrix method (EMM) [25] in the literature. Accordingly, we observe that the presented method is more effective than EMM [25]. As a result, we conclude that the Lerch collocation method is successful and effective.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Degenerate trigonometric functions arising from p-adic integrals on $\mathbb{Z}_p$

Hye Kyung Kim

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$.

The p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\ d\mu(x) &= \lim_{N\to\infty} \sum_{x=0}^{P^N-1} f(x)\mu(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),\end{aligned} \tag{1}$$

(*cf.* [1, 3, 4, 7, 12, 13, 14]).

From (1), we note that

$$\int_{\mathbb{Z}_p} f(x+1)\ d\mu(x) - \int_{\mathbb{Z}_p} f(x)\ d\mu(x) = f'(0), \tag{2}$$

where $f(x)$ is uniform differentiable function on $\mathbb{Z}_p$ (*cf.* [4, 7, 12]).
In [3], the $fermionic$ p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\ d\mu_{-1}(x) \quad &= \lim_{N\to\infty} \textstyle\sum_{x=0}^{p^N-1} f(x)\ \mu_{-1}(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \textstyle\sum_{x=0}^{p^N-1} (-1)^x f(x).\end{aligned} \tag{3}$$

Thus, by (3), we get

$$\int_{\mathbb{Z}_p} f(x+1)\ d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)\ d\mu_{-1}(x) = 2f(0) \tag{4}$$

(*cf.* [1, 3, 13, 14]).

It is well known that the Euler formula is given by

$$e^{ix} = \cos x + i \sin x, \quad (x \in \mathbb{R}, i = \sqrt{-1}), \tag{5}$$

(*cf.* [9]).

From (5), we have

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}. \tag{6}$$

Recently, many mathematicians have studied degenerate versions of some special polynomials and numbers that can take into account the surrounding environment or a person's psychological burden (see [2, 5], [6]-[10]). In this manuscript, we explore the topic of degenerate trigonometric functions that arise from p-adic integrals on $\mathbb{Z}_p$. We derive several interesting identities for degenerate trigonometric functions and degenerate special numbers, which are

obtained from p-adic integral on $\mathbb{Z}_p$. In detail, we establish identities for degenerate Bernoulli numbers, degenerate cotangent numbers, degenerate Euler numbers and degenerate type 2 Euler numbers in relation to degenerate trigonometric functions arising from p-adic integral representations on $\mathbb{Z}_p$. Moreover, we derive explicit formulas for the degenerate tangent numbers and degenerate secant numbers. These identities provide valuable insights into the connections between degenerate trigonometric functions and the degenerate special numbers.

KEYWORDS: Trigonometric functions, p-adic integrals on $\mathbb{Z}_p$, Euler's zigzag number, Tangent numbers, Degenerate Bernoulli and Euler numbers

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Closed-form evaluation of some series involving the Dirichlet characters and the numbers counting Lyndon words

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The main aim of this study is to investigate some infinite series involving the Dirichlet characters and the numbers counting Lyndon words.

2020 MSC: 05A15, 11A25, 11B83, 11M35, 11M36, 11S40, 68R15

KEYWORDS: Arithmetical functions, Dirichlet characters, Dirichlet L-series, Divisor sum, Euler's totient function, Generalized Apostol-Bernoulli numbers, Generating functions, Lyndon words, Möbius function

Introduction

It is quite important to obtain a closed-form expression that can be used instead so that the series can be processed and evaluated easily. In this context, within this study we handle some infinite series, involving the Dirichlet characters and the numbers counting Lyndon words, for the purpose to derive closed-form expressions for this type series.

Some preliminaries regarding the Dirichlet characters and the Dirichlet L-series

Throughout this study, the mostly used concepts are the Dirichlet characters and the Dirichlet L-series in order to achieve the purpose mentioned above. Therefore, we begin by recalling some notations and definitions regarding these concepts with the following brief introduction:

Let $\varphi(n)$ denote the Euler's totient function defined by

$$\varphi(n) = \sum_{\substack{m=1 \\ \mathtt{gcd}(m,n)=1}}^{n} 1,$$

where $\mathtt{gcd}$ denotes the usual greatest common divisor. Observe that the above sum runs over the positive integers that are less than n and relatively prime to n (*cf.* [2, p. 25]).

Let $(\mathbb{Z}/f\mathbb{Z})^*$ denote the group of reduced residue class modulo $f \in \mathbb{N}$. Let $\mathbb{C}\setminus\{0\}$ denote the set of non-zero complex numbers. Then, the rth Dirichlet character with conductor f, denoted by $\chi_{f,r}$, is a group homomorphism from $(\mathbb{Z}/f\mathbb{Z})^*$ to $\mathbb{C}\setminus\{0\}$, i.e. (*cf.* [2]):

$$\chi_{f,r} : (\mathbb{Z}/f\mathbb{Z})^* \to \mathbb{C}\setminus\{0\} \tag{1}$$

where $r \in \{1, 2, \ldots, \varphi(f)\}$, due to the fact that "*there are $\varphi(f)$ distinct Dirichlet characters with modulo f, each of which is completely multiplicative and periodic with period f*" as stated by Apostol [2, Theorem 6.15, p. 138].

To present a few examples, two separate Dirichlet characters with conductor $f = 6$ exist in total, because of the fact that $\varphi(6) = 2$. In other words, we have $\chi_{6,1}$ and $\chi_{6,2}$, which are respectively given in the rows of Table 1 whose entries are obtained by the Implementation 1.

Implementation 1: The following code, written in Wolfram language, returns the entries of Table 1. See, for details, the documentations supplied by [37].

```
Table[ DirichletCharacter [6, j ,n], {j, 1, EulerPhi[6]}, {n, 0, 5}].
```

n	0	1	2	3	4	5
$\chi_{6,1}(n)$	0	1	0	0	0	1
$\chi_{6,2}(n)$	0	1	0	0	0	-1

Table 1: All the Dirichlet characters with conductor $f = 6$ (*cf.* [32])

In addition, because of the fact that $\varphi(7) = 6$, six separate Dirichlet characters with conductor $f = 7$ exist in total. That is, we have $\chi_{7,1}$, $\chi_{7,2}$, $\chi_{7,3}$, $\chi_{7,4}$, $\chi_{7,5}$ and $\chi_{7,6}$, which are also respectively given in the rows of Table 2 whose entries are obtained by the Implementation 2.

Implementation 2: The following code, written in Wolfram language, returns the entries of Table 2. See, for details, the documentations supplied by [37].

```
Table[ DirichletCharacter [7, j ,n], {j, 1, EulerPhi[7]}, {n, 0, 6}].
```

n	0	1	2	3	4	5	6
$\chi_{7,1}(n)$	0	1	1	1	1	1	1
$\chi_{7,2}(n)$	0	1	ω^2	ω	-ω	-ω^2	-1
$\chi_{7,3}(n)$	0	1	-ω	ω^2	ω^2	-ω	1
$\chi_{7,4}(n)$	0	1	1	-1	1	-1	-1
$\chi_{7,5}(n)$	0	1	ω^2	-ω	-ω	ω^2	1
$\chi_{7,6}(n)$	0	1	-ω	-ω^2	ω^2	ω	-1

Table 2: All the Dirichlet characters with conductor $f = 7$, where $\omega = e^{\pi i/3}$ (*cf.* [32])

Further tables demonstrating the Dirichlet characters with conductors $f = 3, 4, \ldots, 8$ can be also found in the studies [2] and [32].

Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then, the Dirichlet L-series associated to $\chi_{f,r}$ is defined as follows:

$$\begin{aligned} L(s, \chi_{f,r}) &= \sum_{n=1}^{\infty} \frac{\chi_{f,r}(n)}{n^s} \\ &= \prod_{p:\ \text{prime}} \left(1 - \frac{\chi_{f,r}(p)}{p^s}\right)^{-1} \end{aligned}$$

which, when $\chi_{f,r} \equiv 1$ (trivial character), reduces to the well-known Riemann zeta function $\zeta(s)$ and its corresponding Euler product, i.e.:

$$\zeta(s) = L(s, 1) = \prod_{p:\ \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

(*cf.* [2, 7, 12, 29, 36, 35]; and the references cited therein).

In recent decades, Dirichlet characters and Dirichlet L-series have been used in the construction and evaluation of many special number and polynomial families (for details, see [4, 10, 31, 13, 32, 30, 29, 26, 28, 27] and the references cited therein).

For instance, the generalized Apostol-Bernoulli numbers $\mathcal{B}_{n,\chi_{f,r}}(\lambda)$, attached to the Dirichlet character $\chi_{f,r}$, are defined by

$$\sum_{j=0}^{f-1} \frac{\chi_{f,r}(j)\lambda^j e^{tj} t}{\lambda^f e^{tf} - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\chi_{f,r}}(\lambda) \frac{t^n}{n!} \tag{2}$$

(*cf.* [1, 8, 9, 10, 11, 33], and the references cited therein).

Note that the numbers $\mathcal{B}_{n,\chi_{f,r}}(\lambda)$ can be written in the following manner:

$$\mathcal{B}_{n,\chi_{f,r}}(\lambda) = f^{n-1} \sum_{j=0}^{f-1} \chi_{f,r}(j)\lambda^j \mathcal{B}_n\left(\frac{j}{f}; \lambda^f\right), \tag{3}$$

where $\mathcal{B}_n\left(\frac{j}{f}; \lambda^f\right)$ denotes the Apostol-Bernoulli polynomials (for details, see [1, 34] and the references cited therein).

If $\chi_{f,r} \equiv 1$ (trivial character) in (2), then the numbers $\mathcal{B}_{n,\chi_{f,r}}(\lambda)$ reduce to the Apostol-Bernoulli numbers. That is, we have

$$\mathcal{B}_n(\lambda) = \mathcal{B}_{n,1}(\lambda)$$

(*cf.* [1, 8, 9, 10, 19, 22, 33] and the references cited therein).

Some preliminaries regarding the Lyndon words and the numbers counting these special words

Throughout this study, the other mostly handled concepts are the Lyndon words and their numbers. Therefore, we continue by recalling some notations and definitions regarding these concepts with the followings:

From the work of Metropolis and Rota [21], it is well-known that a combinatorial necklace composed of n beads coloured with one of k possible colors, and by assuming that coloured beads are represented by letters from an alphabet Σ containing k distinct letters, the rotations on the combinatorial necklace generates some words with n digits. Some of the words generated are aperiodic (primitive) and some are periodic (For example, see Figure 1). A k-ary Lyndon words of length n is known to be identified as the lexicographically (i.e. in a dictionary order) smallest element of an equivalence classes resulting from cyclic shifts of k letters in a primitive word of length n. For details regarding some properties of these special words, glance at [3, 21, 15, 16, 17].

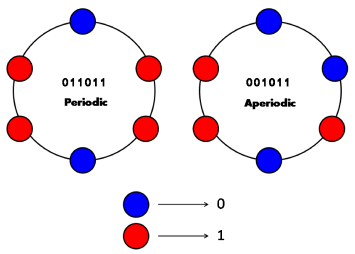

Figure 1: Instances for a periodic and an aperiodic (primitive) necklaces with their representative words (*cf.* [17])

The arithmetical function $L_k(n)$ allows us to count how many k-ary Lyndon words of length n are, and it is given by (*cf.* [3, 5, 21, 18, 20, 23, 24]):

$$L_k(n) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)k^d, \tag{4}$$

where the sum is over all positive divisors of n and μ corresponds the Möbius function defined by (*cf.* [2]):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^m & \text{if } n \text{ is a square-free integer with } m \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ has a squared prime factor.} \end{cases}$$

In [17], Kucukoglu et al. gave a closed-form expression for the following infinite series involving the numbers of the k-ary Lyndon words having prime length p:

$$\frac{\mathcal{B}_2(\lambda)}{2p} - \frac{\mathcal{B}_{p+1}(\lambda)}{p(p+1)} = \sum_{k=1}^{\infty} L_k(p)\lambda^k; \qquad |\lambda| < 1 \tag{5}$$

where $\mathcal{B}_p(\lambda)$ denotes the Apostol-Bernoulli numbers which are rational functions of parameter λ (*cf.* [17]).

For the infinite series involving the numbers of the k-ary Lyndon words having length $n \in \mathbb{Z}^+$, Kucukoglu et al. [16] also gave the following formula:

$$-\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\frac{\mathcal{B}_{d+1}(0;\lambda)}{d+1} = \sum_{k=0}^{\infty} L_k(n)\lambda^k; \qquad |\lambda| < 1$$

where $\mathcal{B}_n(x;\lambda)$ corresponds to the Apostol-Bernoulli polynomials (*cf.* [16]).

In [14], Kucukoglu and Simsek also handled the following alternating series involving the numbers of k-ary Lyndon words having prime length p:

$$f_{L_E}(\lambda, p) = \sum_{k=0}^{\infty}(-1)^k L_k(p)\lambda^k; \qquad |\lambda| < 1$$

and they expressed the above series as the following three different formulas:

$$f_{L_E}(\lambda, p) = \frac{\lambda}{p(1+\lambda)^2}\left(1 - \frac{\lambda+1}{2}\sum_{j=0}^{p-1}\binom{p}{j}\mathcal{E}_j(\lambda)\right),$$

$$f_{L_E}(\lambda, p) = \frac{\lambda}{p(1+\lambda)^2}\left(1 - \sum_{j=0}^{p-1}\binom{p}{j}H_j\left(-\lambda^{-1}\right)\right)$$

and

$$f_{L_E}(\lambda, p) = \frac{\mathcal{E}_p(\lambda)}{2p} + \frac{\lambda}{p(1+\lambda)^2},$$

where $\mathcal{E}_n(\lambda)$ denotes the Apostol-Euler numbers and $H_n(\lambda)$ denotes the Frobenius-Euler numbers (*cf.* [14]).

Moreover, Kucukoglu et al. [17] gave the following formula for another infinite series involving the numbers of the k-ary Lyndon words of length n:

$$\sum_{k=0}^{\infty} L_k(n)\frac{x^k}{k!} = \frac{e^x}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)Bl_d(x), \tag{6}$$

where $Bl_d(x)$ denotes the Bell polynomials (*cf.* [17]).

Main results and concluding remarks

Here, we study on closed-form expressions for some twisted Dirichlet type series involving an arithmetical function counting Lyndon words.

For example, in this study we set

$$F(s, g; \xi) := \sum_{k=0}^{\infty} g(k) \frac{\xi^k}{k^s}, \tag{7}$$

where $g(k)$ is an arithmetical function and ξ is an mth root of 1 such that $\xi \neq 1$ and $m \in \mathbb{N}$.

Substituting $g(k) = \chi_{f,r}(k)$, the infinite series, given by (7), reduces to the twisted L-series which is a special case of the Dirichlet L-series (for details regarding the twisted L-series, see [28, 26, 27, 29] and the references cited therein).

By using the definition of the number of Lyndon words in the series $F(s, g; \xi)$, we especially investigate the fundamental properties of the following series

$$\sum_{k=0}^{\infty} \chi_{f,r}(k) L_k(n) \xi^k \tag{8}$$

which can be expressed in some closed-forms expressions by aid of some arithmetical functions, involving the Dirichlet characters, the Mobius function and the Euler's totient function together with some special numbers and polynomials.

The present study is still in progress, and in the near future, it is planned to obtain closed-form expressions for the series mentioned above and to examine the relationships of these expressions with other number and polynomial families.

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This study has been presented at "The 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023)", which is dedicated to Professor Yilmaz SIMSEK on the Occasion of his 60th Anniversary, held at Université d'Evry in Paris, FRANCE on August 23–27, 2023.

On this occasion, I would like to express my heartfelt thanks and respect to my PhD advisor Professor Yilmaz SIMSEK for his endless supports and efforts to make all his students a good researcher by sharing his valuable knowledge and experiences. For Professor Simsek's great contribution to mathematics and his guidance for young researchers, I would also like to dedicate my current study to his 60th birthday with my best regards. I wish the rest of his life to be happy, fruitful, success and he have many healthy years to spend with his loved ones.

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On Peters-type Simsek numbers and polynomials attached to the Dirichlet character

Irem Kucukoglu

In [16], Simsek introduced a family of numbers and polynomials, which is referred as Peters-type Simsek numbers and polynomials attached to the Dirichlet character. In the chapter [17], Simsek and Kucukoglu investigated these numbers and polynomials by providing new and old results with some applications regarding these numbers and polynomials. In [8], Kucukoglu dealt with the higher-order Peters-type Simsek numbers and polynomials attached to the Dirichlet character and presented numerous tables and plots concerning their distinct special cases. By the current presentation, it is aimed to present a survey and an investigation on some fundamental properties of these numbers and polynomials. By the end of the presentation, the talk will be ended by some comments with the mention of some remarks and open problems which are be able to find many applications in computational science and engineering and related branches, in case they can be solved.

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Keywords: Dirichlet characters, Euler's totient function, Generating functions, Group homomorphism, Peters-type Simsek numbers and polynomials of the first kind, Special numbers and polynomials, Two-variable Peters-type Simsek polynomials

Introduction, preliminaries and notations

Throughout this talk, let the notation $\varphi(f)$ show the Euler's totient function and let the notation $\chi_{f,r}$ indicate a map from the group $(\mathbb{Z}/f\mathbb{Z})^*$ of reduced residue class modulo $f \in \mathbb{N}$ to the set of non-zero complex numbers $\mathbb{C}\setminus\{0\}$, as a group homomorphism referred to the r-th Dirichlet character with conductor f (*cf.* [1]):

$$\chi_{f,r} : (\mathbb{Z}/f\mathbb{Z})^* \to \mathbb{C}\setminus\{0\} \tag{1}$$

such that $r \in \{1, 2, \ldots, \varphi(f)\}$ because of the fact that "*there are $\varphi(f)$ distinct Dirichlet characters with modulo f, each of which is completely multiplicative and periodic with period f*" as stated by Apostol [1, Theorem 6.15, p. 138].

The concept of Dirichlet characters have been recently used as a tool in the construction of generating functions for some special numbers and polynomials, and it still continues to be used. To see some examples, refer to the papers [2, 5, 16, 8, 17, 15, 14, 11, 13, 12] and the references cited therein. Amongst these instances, the Peters-type Simsek numbers and polynomials attached to the Dirichlet character, denoted respectively by $Y_{n,\chi_{f,r}}(\omega, q)$ and $Y_{n,\chi_{f,r}}(x; \omega, q)$, were constructed by Simsek [16] with the following generating functions, for $\omega \in \mathbb{Z}_p$, the set of p-adic integers, with $\omega \neq 1$:

$$\begin{aligned} G_Y(t;\omega,q,\chi_{f,r}) &:= \frac{(1+q)\sum_{j=0}^{f-1}(-1)^j\chi_{f,r}(j)\,(\omega q)^j\,(1+\omega t)^j}{(\omega q)^f\,(1+\omega t)^f-1} \\ &= \sum_{n=0}^{\infty} Y_{n,\chi_{f,r}}(\omega,q)\frac{t^n}{n!} \end{aligned} \tag{2}$$

and

$$\begin{aligned} G_Y(t,x;\omega,q,\chi_{f,r}) &:= (1+\omega t)^x G_Y(t;\omega,q,\chi_{f,r}) \\ &= \sum_{n=0}^{\infty} Y_{n,\chi_{f,r}}(x;\omega,q)\frac{t^n}{n!}, \end{aligned} \tag{3}$$

where $\chi_{f,r}$ denotes the r-th Dirichlet character with conductor $f \in \{2n : n \in \mathbb{N}\}$ such that $r \in \{1,2,\ldots,\varphi(f)\}$ (*cf.* [16]).

If $q \to 1$ and $\chi_{f,r} \equiv 1$ (trivial character) in (2) and (3), the numbers $Y_{n,\chi_{f,r}}(\omega,q)$ and the polynomials $Y_{n,\chi_{f,r}}(x;\omega,q)$ reduce respectively to the Peters-type Simsek numbers $Y_n(\omega)$ and the Peters-type Simsek polynomials $Y_n(x;\omega)$ of the first kind, defined by

$$G_Y(t,\omega) := \frac{2}{\omega(1+\omega t)-1} = \sum_{n=0}^{\infty} Y_n(\omega)\frac{t^n}{n!} \tag{4}$$

and

$$G_Y(t,x,\omega) := (1+\omega t)^x G_Y(t,\omega) = \sum_{n=0}^{\infty} Y_n(x;\omega)\frac{t^n}{n!}, \tag{5}$$

(see, for details, [16]).

To learn further details, properties, tables and plots concerning the Peters-type Simsek numbers and polynomials attached to the Dirichlet character and the Peters-type Simsek numbers and polynomials, refer to the papers [16, 18, 17].

Two-variable Peters-type Simsek polynomials of the first kind, denoted by $Y_n(x,y;\omega,\delta)$, was introduced by Khan et al. [3] with the following generating function:

$$\frac{2(1+\omega t)^x\left(1+\delta t^2\right)^y}{\omega(1+\omega t)-1} = \sum_{n=0}^{\infty} Y_n(x,y;\omega,\delta)\frac{t^n}{n!},$$

such that

$$Y_n(0,0;\omega,\delta) = Y_n(\omega),$$

(*cf.* [3, 4]).

To learn further details, properties, tables and plots concerning the polynomials $Y_n(x,y;\omega,\delta)$, refer to the papers [3, 4].

The higher-order Peters-type Simsek numbers and polynomials attached to the Dirichlet character, denoted respectively by $Y_{n,\chi}^{(k)}(\omega,q)$ and $Y_{n,\chi}^{(k)}(x;\omega,q)$, were introduced by the author [8] with the following generating functions:

$$\begin{aligned} \mathcal{K}_Y\left(t;k,\omega,q,\chi_{f,r}\right) &:= \left(\frac{1+q}{(\omega q)^f\,(1+\omega t)^f-1}\right)^k \sum_{j=0}^{f-1}(-1)^j\chi_{f,r}(j)\,(\omega q)^j\,(1+\omega t)^j \\ &= \sum_{n=0}^{\infty} Y_{n,\chi_{f,r}}^{(k)}(\omega,q)\frac{t^n}{n!} \end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathcal{K}_Y\left(x,t;k,\omega,q,\chi_{f,r}\right) &:= (1+\omega t)^x\mathcal{K}_Y\left(t;k,\omega,q,\chi_{f,r}\right)\\ &= \sum_{n=0}^{\infty}Y_{n,\chi_{f,r}}^{(k)}(x;\omega,q)\frac{t^n}{n!}.\end{aligned} \tag{7}$$

where $\chi_{f,r}$ denotes the r-th Dirichlet character even conductor $f\in\{2n:n\in\mathbb{N}\}$ such that $r\in\{1,2,\ldots,\varphi(f)\}$ (*cf.* [8]).

Observe that if $k=1$ in (6) and (7), the numbers $Y_{n,\chi}^{(k)}(\omega,q)$ and the polynomials $Y_{n,\chi}^{(k)}(x;\omega,q)$ reduce respectively to the Peters-type Simsek numbers and polynomials attached to the Dirichlet character (*cf.* [8]).

Pay attention to the case when we set to be $q\to 1$, $\chi_{f,r}\equiv 1$ (trivial character) and k is a nonnegative integer in (6) and (7), in which the numbers $Y_{n,\chi}^{(k)}(\omega,q)$ and the polynomials $Y_{n,\chi}^{(k)}(x;\omega,q)$ reduce respectively to the positive higher-order Peters-type Simsek numbers and polynomials (*cf.* [6]). On the other hand, if we set to be $q\to 1$, $\chi_{f,r}\equiv 1$ (trivial character) and k is a negative integer in (6) and (7), the numbers $Y_{n,\chi}^{(k)}(\omega,q)$ and the polynomials $Y_{n,\chi}^{(k)}(x;\omega,q)$ reduce respectively to the negative higher-order Peters-type Simsek numbers and polynomials (*cf.* [7]).

To learn further details, properties, tables and plots concerning the higher-order Peters-type Simsek numbers and polynomials attached to the Dirichlet character, refer to the papers [8]. For the positive and negative higher-order Peters-type Simsek numbers and polynomials, see also [6, 7].

The multiparametric higher-order Hermite-based Peters-type Simsek polynomials of the first kind, denoted by ${}_HY_n^{(k)}\left(x,\overrightarrow{y},z,\omega,m\right)$, introduced and investigated by the author in [9] and [10] with the following generating function:

$$\begin{aligned}F_H\left(t,x,\overrightarrow{y},z,\omega,m,k\right) &:= \left(\frac{2}{\omega\left(1+\omega t\right)-1}\right)^k(1+\omega t)^x\exp\left(zt+\sum_{j=1}^{m}y_jt^j\right)\\ &= \sum_{n=0}^{\infty}{}_HY_n^{(k)}\left(x,\overrightarrow{y},z,\omega,m\right)\frac{t^n}{n!},\end{aligned} \tag{8}$$

where exp denotes the natural exponential function, m is a positive integer, k is a nonnegative integer, ω is an arbitrary (real or complex) parameter, $\overrightarrow{y}=(y_1,y_2,\ldots,y_m)$ is an m-tuple of real numbers, $z=(\alpha,\beta):=\alpha+i\beta$ is a complex number so that α and β are real numbers and $i^2=-1$ (*cf.* [9, 10]).

To learn further details, properties, tables and plots concerning the multiparametric higher-order Hermite-based Peters-type Simsek polynomials of the first kind, refer to the papers [9, 10].

Overview of the presentation and concluding remarks

In this presentation, in line with the definitions given in the previous section, we present a survey and an investigation on some fundamental properties of the Peters-type Simsek numbers and polynomials with their improvements and generalizations proposed by some researchers.

Our future plan is to handle the remarks and open questions, which were posed by the author in [8], concerning the Peters-type Simsek numbers and polynomials attached to any group homomorphism.

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This study has been presented at "The 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2023)", which is dedicated to Professor Yilmaz SIMSEK on the Occasion of his 60th Anniversary, held at Université d'Evry in Paris, FRANCE on August 23–27, 2023.

On this occasion, I would like to express my heartfelt thanks and respect to my PhD advisor Professor Yilmaz SIMSEK for his endless supports and efforts to make all his students a good researcher by sharing his valuable knowledge and experiences. For Professor Simsek's great contribution to mathematics and his guidance for young researchers, I would also like to dedicate my current study to his 60th birthday with my best regards. I wish the rest of his life to be happy, fruitful, success and he have many healthy years to spend with his loved ones.

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On $((p,q),\omega)$-Sturm-Liouville problem and their orthogonal solutions

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In this study, the $((p,q),\omega)-$derivative and $((p,q),\omega)-$integration are investigated and we called that new class of calculus as a $(p,q)-$Hahn Calculus. We consider the new class of Sturm-Liouville problems namely $((p,q),\omega)-$Sturm-Liouville problem, and demonstrates that their solutions are orthogonal concerning $((p,q),\omega)$-integral space. We derives the $((p,q),\omega)$ hypergeometric expression for the polynomial solutions, along with their corresponding three-term recurrence relations.

2020 MSC: 33C47, 42C05

KEYWORDS: $((p,q)$,ω-Sturm-Liouville problems, ((p,q)ω)-Pearson equation, $((p,q)$,$\omega)$ orthogonal solutions, $((p,q)$,$\omega)$-difference operator

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Data minimization in scientific research

Imge Ozer

Data Minimization, which is one of the seven basic data protection principles within the scope of the General Data Protection Regulation (GDPR), is a principle that states that the data processed to ensure data privacy should not be kept, transferred, or used unless it is necessary for reasons clearly stated beforehand. In other words, according to this principle, data should be collected and processed in a way that is sufficient, relevant, and limited to the purpose for which they are processed. Data minimization aims to prevent risks and damages regarding data security as the amount and variety of personal data increases. For this reason, non-purpose and non-obligatory data should never be collected, and data that will no longer be used should be deleted or anonymized/pseudonymized immediately. Just like in the social sciences, the processing of statistics containing personal data in applied mathematics or the examination of the effects on the subject investigated in geological research on the health of individuals should be carried out by the general principles indicated in the GDPR. In this context, all scientific research activities should be based on a legal basis for data processing activities. Of course, there are certain exceptions for personal data to be processed within the scope of scientific research under the GDPR. However, to keep personal data longer than necessary for the purpose for which it was processed, i.e., archiving in the public interest, scientific or historical research, and statistical purposes, an additional compatibility test is required to reuse that personal data. Apart from this, the "Research Protocol," "Analysis Plan," etc. to be signed between researchers and data subjects. While these documents determine the methodological standards, "Informed Consent," "Investigator Agreement," and "Ethics Committee Approval," etc., documents form the legal basis.

2020 MSC: 62H10, 62H05

KEYWORDS: Personal data protection, Data minimization, Data, Privacy, Research

Introduction

Scientific research enables the cumulative progress of knowledge and creates large masses of datasets to reach research results. It is called "Good data" if these piles are classified by grouping them in a controlled manner, or if it is stored in a complex form without classification, we encounter bad data combinations. In this sense, we need legal regulations to make the necessary classifications. How to interpret the research-related provisions in GDPR and British legislation is explained in the guide [1] on the evaluation of scientific publications published by the UK Data Protection Authority ICO for data protection officers and DPO's in institutions and organizations that conduct research, archiving, or processing for statistical purposes within the scope of GDPR. Research provisions refer to three research-related goals: Archiving intents for the public interest, Scientific research, and Statistical purposes. For the requirements regarding the research to be applied, it must be proven that the data processing activity is necessary for at least one of these purposes. The most basic criterion when determining whether data processing can be evaluated within the scope of research provisions is the purpose of performing data processing. Scientific research, developing the latest technology in a particular field or offering innovative solutions to problems; generating new insights or insights that contribute to human knowledge in a given area; supporting education; It aims to produce general practice findings for testing and replication. Scientific research mainly includes the basis undertaken to acquire new knowledge, experimental or theoretical work, knowledge of the underlying foundations of phenomena and observable realities, application or use in any particular aspect, applied research, and original research undertaken to acquire new knowledge. While processing personal data, care should be taken to ensure that the data is linked, limited, and proportional to the purpose for which they are processed [2]. Data should only be processed if another method fails to fulfill the processing purpose. For example, the data obtained by saying it can be used in the future creates "data confusion" and expresses the opposite of this principle. Minimizing the number of data processed with this method provides excellent convenience in determining whether the data processing is done according to the basic principles. The primary purpose of the principle is to encourage the use of other tools that interfere less with the fundamental rights and freedoms of the data subject if the purpose of processing the relevant data can be achieved. Data controllers must first determine why they need it and whether they have enough data to process. For this reason, it is necessary to question whether the data processed periodically is sufficient, vital, and relevant. Where some data is out of sufficiency, necessity, or relevance, four essential methods have been proposed to ensure data minimization under GDPR. These are privacy by design, privacy by default, anonymization, pseudonymization, and accountability. According to the principle of privacy by design, data controllers should redesign all the tools of the system they use while processing personal data and take measures to protect them from the very beginning. In privacy by default, privacy is fulfilled in the strictest manner determined by data controllers from the beginning of the processing activity to the extent permitted by law. In anonymization, all information that directly or indirectly identifies the data subject in a data collection is used by losing its distinctiveness within a group so that it cannot be associated with it [3]. The masking method is applied in pseudonymization by giving an alias instead of the data subject. It differs from anonymization in that it does not eliminate the possibility of associating this data with natural persons [4]. Accountability, on the other hand, is the burden of proof that data controllers act by the law in broad scope, from informing the data subjects about all these methods to preventing the illegal processing of personal data and access to personal data to obtaining their explicit consent under certain conditions, and to deleting/destroying this data [5].

Materials and method

The content of this research is aimed to be theoretically focused while being supported by national and international practices. In this direction, the method determined inwriting the paper consists of three basic stages:

1. Scanning international and foreign legal doctrines and sources through various sources, especially electronic databases,

2. Scanning the existing literature, especially the main printed sources, in the field of personal data protection law,

3. Analyzing the outputs obtained from international, regional and foreign legal sources and doctrine, especially the principles set forth by international treaties, on a subject-specific basis. Due to the limited number of sources in Turkish law on the subject, the research was mainly based on sources written in English. Methods such as empirical or sociological research were not used in the study, and the study was continued by analyzing the inferences reached as a result of systematic readings carried out on the compiled sources on the legal basis.

Results

Data minimization is also essential for scientific research, especially involving personal or sensitive data. Data minimization can help researchers protect the privacy and confidentiality of their participants and comply with ethical and legal requirements. Data minimization can also improve the quality and validity of the research results by reducing the noise and bias that may arise from collecting unnecessary or irrelevant data [6]. Some of the ways that researchers can apply data minimization in their projects are:

- Defining the research question and objectives clearly and precisely, and identify the minimum data needed to answer them.
- Using existing data sources or secondary data whenever possible instead of collecting new data from scratch.
- Using anonymization or pseudonymization techniques to remove or replace any identifying information from the data, such as names, addresses, phone numbers, etc.
- Using aggregation or summarization methods to reduce the granularity or detail of the data, such as averages, ranges, categories, etc.
- Using sampling or filtering techniques to select only a subset of the data that are relevant and representative for the research purpose, such as random sampling, stratified sampling, etc.
- Delete or destroy any data that are no longer needed for the research purpose or after a certain period [7].

Discussion and conclusions

Data minimization is a legal obligation and a good practice for scientific research. It can help researchers respect the rights and interests of their participants and enhance the reliability and efficiency of their research process. The application of personal data protection legislation in scientific researches is also considered an ethical problem. The data subject's consent is not always required to process personal data. Still, it is ethically relevant to obtain the consent of the data subjects for the protection of them in cases where an exception is made for data processing for scientific research purposes, provided that it does not violate public safety, privacy, or personal rights. Ethical considerations in scientific research are principles that guide research designs and practices. Scientists and researchers must always adhere to a specific code of conduct when collecting data from people. These principles include:

- Voluntary participation: Research participants should freely agree to participate in the study without coercion or pressure. They should also be able to withdraw from the study without any negative consequences [8].
- Informed consent: Research participants should be fully informed about the purpose, methods, risks, benefits, and alternatives of the study before they agree to participate. They should also be allowed to ask questions and clarify any doubts [9].
- Anonymity: Research participants should not be identified by name or other personal information that could reveal their identity. Their data should be coded or encrypted to protect their privacy.
- Confidentiality: Research participants' data should be stored securely and accessed only by authorized persons. Their data should not be shared or disclosed to anyone without their permission unless required by law.
- Potential for harm: Research participants should not be exposed to any physical, psychological, social, or legal damage due to their participation in the study. Any potential harm should be minimized or avoided as much as possible.
- Results communication: Research participants should be informed about the results and implications of the study, as well as their right to access their data. The researcher should also disseminate the findings of the survey in an honest, accurate, and transparent manner while respecting the confidentiality and anonymity of the participants [10].

These ethical considerations are common across scientific disciplines and international borders. However, they may vary slightly depending on the research's specific context, nature, and scope. Therefore, researchers should always consult the relevant ethical guidelines and personal data protection regulations before conducting their research. Other legal grounds for scientific research may be the fulfillment of a contractual provision, the completion of a legal obligation, the public interest, or legitimate interests pursued by the data controller or a third party (GDPR art. 6; recital 47).

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On a sequence of positive linear operators related to squared-Durrmeyer operators

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The aim of this paper is to study the pointwise behavior of certain sequences of the squared-Durrmeyer operators DB_n^2 and $\widetilde{DB}_n^2$ acting on bounded functions on an interval $[0,1]$, defined by Gavrea and Ivan. Here we estimate the rate of convergence at a point y, which is a Lebesgue point of $g \in L_1([0,1])$ be such that $\psi o\,|g| \in BV([0,1])$, where $\psi o\,|g|$ denotes the composition of the functions ψ and $|g|$. The function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$, which appears from the $(L-\psi)$ Lipschitz conditions.

2020 MSC: 41A35, 41A25, 47G10

KEYWORDS: Squared-Durrmeyer operators, Bounded variation, Lipschitz condition, Pointwise convergence

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Mathematical modeling and numerical simulation on tornado dynamics

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A tornado features combining three major co-existent kinematic components, namely, updraft, translation, and rotation, which involve all three dimensions in space while transient in time. For numerical simulation of a tornado-building interaction scenario, it looks quite challenging to secure a set of physically-rational and meanwhile computationally-practical boundary conditions to accompany traditional Computational Fluid Dynamics (CFD) approaches and, as of today, little literature can be found on the three-dimensional (3D) computational tornado dynamics study. Inspired by the development of immersed boundary (IB) method, this study started with re-tailoring the Rankine-combined vortex model (RCVM) by applying the "*relative motion*" principle to the translational component of tornado, such that the building is viewed as "*virtually*" translating towards a "*pinned*" rotational flow that remains time-invariant at the far field region. This revision renders a steady-state kinematic condition applicable to the outer boundary of a large tornado simulation domain, successfully circumventing the boundary condition updating process that the original RCVM would have to undergo, and tremendously accelerating the computation. Furthermore, this re-tailored RCVM was extended to its 3D version with the aid of logarithm law for describing the vertical turbulent flow variation at the boundary of 3D computational domain. This 3D tornado model has been embedded in Incompact3D, an academic high-order finite difference turbulent flow solver, resulting in a practical 3D tornado-building interaction simulation tool. A case study examined the tornadic wind induced loadings on a prismatic building. Over all three directions, the vertical force component was found dominant, which effectively echoes the uprooting effect as observed in the reported aftermath of devastating tornadoes.

2020 MSC: 76D05, 35R37, 65L12, 76F65

KEYWORDS: Re-tailored Rankine-combined vortex model (RCVM), Immersed boundary (IB), Logarithm law, Finite difference method, Wind loadings

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Catalan, Fuss-Catalan and Raney numbers: Generic combinatorial interpretation and extended definition

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The Catalan sequence is a sequence of nonnegative integers that is, in many different contexts, the answer to the question "how many elements of size n are there, given any $n \in \mathbb{N}$?".
The Fuss-Catalan sequences answer the same type of question, they are a generalization of the Catalan sequence, as the Catalan sequence is the Fuss-Catalan sequence of parameter 2.
The same goes for Raney sequences, which are a two-parameter generalization of the Fuss-Catalan sequences, as the Fuss-Catalan sequence of parameter p is the Raney sequence of parameters $(p, 1)$.
Here, we focus on what links the different contexts in which such a sequence is the solution, and show a purely algebraic proof for their explicit formula, i.e. without using any complex analysis. The three main results shown here are :

- Sufficient conditions on a combinatorial class such that the sequence (indexed by n) of the number of elements of size n from the class is a Fuss-Catalan or Raney sequence.
- An explicit construction of an isomorphism between any two combinatorial classes that verify these conditions, with a proof that isomorphisms constructed this way are closed under composition.
- A new approach of Raney numbers with any parameters (not necessarily integers), in a commutative ring that contains $\mathbb{Q}$, which is just enough of a structure to define binomial coefficients properly, and proofs of all the usual Raney numbers' identities in this context.

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KEYWORDS: Catalan numbers, Fuss-Catalan numbers, Raney numbers

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Remarks on degenerate Simsek numbers

Lahcen Oussi

In 2018, Y. Simsek [14] investigated a new family of special numbers by means of the following generating function

$$F_{Y_1}(t,k;\lambda) := \frac{(\lambda e^t+1)^k}{k!} = \sum_{n\geq 0} Y_1(n,k;\lambda)\frac{t^n}{n!}. \tag{1}$$

These numbers allow computing negative order Euler numbers and related numbers and polynomials. They are related to the well-known special numbers, such as Stirling numbers of the second kind, Bernoulli numbers, Lucas numbers, Fibonacci numbers, etc.

The Simsek numbers $Y_1(n,k;\lambda)$ can be expressed by the following explicit formula

$$Y_1(n,k;\lambda) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i} i^n \lambda^i. \tag{2}$$

Setting $\lambda = 1$ in Eq. (2), we obtain the following well-known numbers (*cf.* [14])

$$B(n,k) := k!Y_1(n,k;1) = \sum_{i=0}^{n}\binom{k}{i} i^n, \tag{3}$$

which are related to the numbers of the form $b_k 2^k$, where b_k is a sequence of positive integers. For more details about the numbers $B(n,k)$ and their combinatorial applications we refer the readers to [2, 12, 14, 15].

In recent years, obtaining degenerate versions of special numbers and polynomials has been a very popular subject of many mathematicians (*cf.* [1, 4, 6], [8]-[13]). In particular, the degenerate Simsek numbers were studied by the named author [12]. For $k \in \mathbb{N}_0 := \{0,1,2,3,\ldots\}, \lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}\setminus\{0\}$, the degenerate Simsek numbers can be defined by means of the following generating function

$$F_{Y_1}(t,k;\lambda|\alpha) := \frac{1}{k!}(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}}+1)^k = \sum_{n\geq 0} Y_1(n,k;\lambda|\alpha)\frac{t^n}{n!}. \tag{4}$$

Obviously, for letting $\alpha \to 0$ in the above identity, we arrive at (2). The explicit formula for the degenerate Simsek numbers is given as follows (*cf.* [12]):

$$Y_1(n,k;\lambda|\alpha) = \frac{1}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}\binom{k}{j} j^m \lambda^j \alpha^{n-m} s(n,m) \tag{5}$$

or

$$Y_1(n,k;\lambda|\alpha) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\alpha^n\lambda^j\left(\frac{j}{\alpha}\right)_n, \tag{6}$$

where $s(n,m)$ denote the Stirling numbers of the first kind defined by means of the generating function

$$\frac{(\log(1+t))^k}{k!} = \sum_{n\geq k} s(n,k)\frac{t^n}{n!},$$

and $(x)_n$ denotes the falling factorial given by

$$\begin{cases} (x)_n = \prod_{j=0}^{n-1}(x-j), & \text{for } n \geq 1; \\ (x)_0 = 1. \end{cases} \tag{7}$$

Here, for $z \in \mathbb{C}$, we assume that $\log z$ denotes the principal branch for the many-valued function with the imaginary part $\mathrm{Im}(\log z)$, constrained by

$$-\pi < \mathrm{Im}(\log z) \leq \pi.$$

Some special cases of the degenerate Simsek numbers are given as follows:

- $y_1(0,k;\lambda|\alpha) = \frac{(\lambda+1)^k}{k!}$
- $y_1(n,0;\lambda|\alpha) = \delta_{n,0}$
- $y_1(n,1;\lambda|\alpha) = \delta_{n,0} + \lambda\left(\frac{1}{\alpha}\right)_n \alpha^n$

The degenerate Simsek numbers $Y_1(n,k;\lambda|\alpha)$ can be also expressed explicitly by

$$Y_1(n,k;\lambda|\alpha) = \frac{1}{k!}\sum_{j=0}^{n}\binom{k}{j} j!\lambda^j(\lambda+1)^{k-j}S_2(n,j|\alpha), \tag{8}$$

(*cf.* [12]), where $S_2(n,k|\alpha)$ denote the degenerate Stirling numbers of the second kind introduced by D. S. Kim and T. Kim [5] by means of the generating function

$$\frac{(e^{\frac{\log(1+\alpha t)}{\alpha}}-1)^k}{k!} = \sum_{n\geq 0} S_2(n,k|\alpha)\frac{t^n}{n!}.$$

Moreover, the degenerate Simsek numbers $Y_1(n,k;\lambda|\alpha)$ satisfy the following recursive formula [12]:

$$Y_1(n+1,k;\lambda|\alpha) = (k-\alpha n)Y_1(n,k;\lambda|\alpha) - Y_1(n,k-1;\lambda|\alpha).$$

It is worth mentioning that the degenerate Simsek numbers $Y_1(n,k;\lambda|\alpha)$ are related to some kinds of special numbers. For instance, specializing the parameters λ and α in $Y_1(n,k;\lambda|\alpha)$, we get Stirling numbers of the second kind, Simsek numbers [14], the λ-Stirling numbers of the second kind [16, 7], the weighted Stirling numbers of the second kind [3], etc.

2020 MSC: 11B73, 11B75, 11B83

KEYWORDS: Degenerate Simsek numbers, Generating function, Stirling numbers

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Spatial data case: Conditional hazard estimate by the local linear method

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The purpose of the present paper is to investigate by the local linear method a nonparametric estimator of the conditional hazard of scalar response variable given a functional variable when the observations are spatially dependent. The main goal is to establish the almost complete convergence with rate of this estimator under some general conditions. A practical example on the climatological data shows the usefulness of our theoretical study.

KEYWORDS: Spatial functional data, Local linear estimation, Conditional hazard function, Strongly mixing process

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Palindrome composition sets and the patterns

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In this paper, we focus on the palindrome composition of an integer. We find the generating function for a numbers of palindrome composition of an integer. Finally, we make pattern palindrome color composition of the positive integers with respect to coloring rule.

2020 MSC: 05A15, 05A17, 05A18, 11B39, 11B99

KEYWORDS: Compositions of the integers, The n-color combination of the integers, Palindromes compositions, Generating function

Introduction

Partitions and compositions of an integer have been the focus of attention of many scientists from past to present. Due to its contributions to many branches of science, the theory of partition has begun to be more up-to-date by studying more and more.

In recent years, many scientists (*cf.* [1, 2, 3, 12, 21, 22, 24]) have studied n-color compositions of an integer m defined as the composition of m for which a part of size n can take on n colors. There are twenty one $n-$color compositions of 4:

$$(4_1), (4_2), (4_3), (4_4), (1_1, 3_1,), (1_1, 3_2), (1_1, 3_3), (3_1, 1_1), (3_2, 1_1), (3_3, 1_1), (2_1, 2_1),$$

$$(2_1, 2_2), (2_2, 2_1), (2_2, 2_2), (2_1, 1_1, 1_1), (1_1, 2_1, 1_1), (1_1, 1_1, 2_1), (2_2, 1_1, 1_1),$$

$$(1_1, 2_2, 1_1), (1_1, 1_1, 2_2), (1_1, 1_1, 1_1, 1_1).$$

When we assign the color for each parts like the following Figure 1;

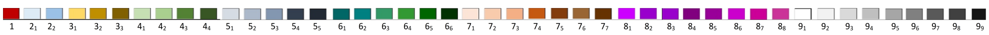

Figure 1: The charts of the color in the color compositions

We can represent the n-color compositions of 4 in rectangles with dimension 4×1 as in Figure 2;

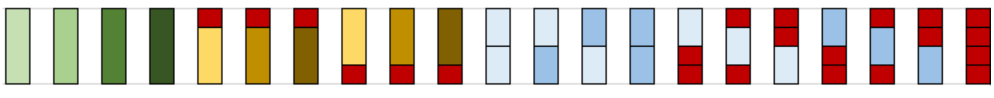

Figure 2: The $n-$color compositions of 4

The similar problem for partitions has been studied to some content under the pen-name “n copies of n” (*cf.* [2, 23]).

Work has also been done on compositions with somewhat constrained piece sizes, and recent progress has been made for n-color compositions.

In [12], Guo taked care of in n-color compositions with no parts of size1. In these works, they are careful to the size of parts in n-color compositions. Also Agarwal (2000), studied $n-$color compositions with no parts greater than k. Then Shapcott (2012) derived results in a more efficient. Shappcott in this paper, ratherthan requiring a part of size n to take on exactly n colors, let a part of size n to take on $c_n \in Z^+$ colors, and rather than forbidding a single part-size or interval of part-sizes, let any selection of part-sizes to be forbidden.

Theorem 1 (*cf.* [24, Theorem 1.1, p. 297])**.** *The number of n-color compositions of a positive integer m is equal to the sum of the part-products over all compositions of m.*

Proof. For a composition $a = \{a_1, a_2, a_3, ...\}$. Let P_m be the set of all ordinary compositions of m and let C_m denote the set of all n-color compositions of m. The number of ways to construct an n-color composition having t parts of size $a_1, a_2, a_3, ...$ is equal to $\prod_{i=1}^{t} a_i$ since a part of size a_i can take on a_i colors. Each choice of part-sizes corresponds to exactly one composition. Hence,

$$|C_m| = \sum_{a \in P_m} \prod_{i=1}^{t} a_i.$$

□

In [1, 13], showed that the number of n-color compositions of m is also F_{2m}.

A palindrome composition of the positive integer m is a composition whose part-sequence is the same whether it is read from left to right to right to left. In [14], it is shown that are $2^{\|\frac{m}{2}\|}$ palindromes compositions of m and also in [24], Shapcott investigated the formula for the numbers of n-color palindrome compositions of a positive integer.

Example 2. *Respectively palindrome compositions of* 5 *and* 6*,*

5	6
131	33
212	222
11111	141
	2112
	1221
	11211
	111111.

n-color palindrome compositions

In this section, we also deal with the palindrome color in the palindrome compositions and we investigate the generating function for its numbers. Moreover we make the pattern of the n-color palindrome compositions of a positive integer with respect to the palindrome color.

Now our aim is to investigate the generating function for the numbers of color palindrome compositions and so we need the following notations for the general form;

From [6], let $\chi(x)$ be the generating function for the numbers showing the change of the axis in the palindrome type, i.e. $\chi(t) = \sum_{i=0}^{\infty} a_i t^i$.

Let $P(t)$ be the generating function for the numbers showing the change of the wings in a palindrome type, i.e. $P(t) = 1 + \sum\limits_{i=1}^{\infty} b_n t^n$ and so b_n is the numbers of the wings in a palindrome type of the integer n.

We need to revise the functions $\chi(t)$ and $P(t)$ to obtain the general form of the generating functions for the numbers of the palindrome type as the following;

$$\begin{array}{rr} Pb(t) = \sum\limits_{i=0}^{\infty} b_i t^{2i}, & \chi b(t) = \sum\limits_{i=1}^{\infty} a_i t^{2i}, \\ \chi o(t) = \sum\limits_{i=0}^{\infty} a_{2i+1} t^{2i}, & \chi e(t) = \sum\limits_{i=1}^{\infty} a_{2i} t^{2i}. \end{array}$$

Then by the Cauchy products of series, we get the following products

$$\begin{aligned} \chi o(t)Pb(t) &= \sum_{i=0}^{\infty} a_{2i+1}t^{2i} \sum_{i=0}^{\infty} b_i t^{2i} = \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} a_{2i+1} b_{m-i} \right) t^{2m}, \\ \chi e(t)Pb(t) &= \sum_{i=1}^{\infty} a_{2i}t^{2i} \sum_{i=0}^{\infty} b_i t^{2i} = \sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} a_{2i} b_{n-i} \right) t^{2m}, \\ \chi b(t)Pb(t) &= \sum_{i=1}^{\infty} a_i t^{2i} \sum_{i=0}^{\infty} b_i t^{2i} = \sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} a_i b_{n-i} \right) t^{2m}. \end{aligned}$$

Now we are ready to find the generating function for palindromes compositions with respect to $\chi(t)$ and $P(t)$, which will be used frequently.

Theorem 3 (*cf.* [6])**.** *The generating function for palindromes compositions with respect to above generating functions is*

$$pal(t) = Pb(t) \left[\chi(t) + \chi b(t) \right].$$

Proof. Let $pal(t) = \sum_{n=1}^{\infty} \alpha_n t^n$ be a generating function for palindromes compositions. Then when we separate $pal(t)$ within even and odd index, we we have the following identities;

$$\begin{aligned} pal(t) &= \sum_{n=1}^{\infty} \alpha_{2n} t^{2n} + t \sum_{n=0}^{\infty} \alpha_{2n+1} t^{2n} \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_{2i} b_{m-i} + \sum_{j=1}^{m} a_j b_{m-j} \right) t^{2m} + t \sum_{m=0}^{\infty} (\sum_{i=0}^{m} a_{2i+1} b_{m-i}) t^{2m} \\ &= \chi e(t)Pb(t) + \chi b(t)Pb(t) + t \left[\chi o(t)Pb(t) \right] \\ &= \chi e(t)Pb(t) + \chi b(t)Pb(t) + t \left[\chi o(t)Pb(t) \right] \\ &= Pb(t) \left[\chi e(t) + t\chi o(t) + \chi b(t) \right] \\ &= Pb(t) \left[\chi(t) + \chi b(t) \right]. \end{aligned}$$

Then we have completed the proof. □

Theorem 4 (*cf.* [6])**.** *The generating function for the numbers of the n-color palindrome compositions is*

$$cp_1(t) = \frac{t^3 + 3t^2 + t}{t^4 - 3t^2 + 1} = t + 3t^2 + 4t^3 + 9t^4 + 11t^5 + 24t^6 + 29t^7 + 63t^8 + 76t^9 + ...$$

Proof. To compute the generating function for the numbers of the n-color palindrome compositions, we need to determine the generating functions for the numbers of the wings in the n-color palindrome compositions and the numbers showing the change of the axis in the n-color palindrome composition $\chi(t)$ by Theorem 3.

By the definition of the n-color palindrome composition, the axis changes with respect to positive integer and so the generating function for the numbers showing the change of the axis in the n-color palindrome composition is

$$\chi(t) = \frac{t}{(1-t)^2} = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 6t^6 + 7t^7 + 8t^8 ...$$

On the other hand, the generating function for the numbers of the wings in the n-color palindrome compositions is ones of the even Fibonacci numbers. Then it follows that

$$Pb(t) = 1 + \frac{t^2}{1 - 3t^2 + t^4}.$$

By Theorem 3, we compute the generating function for the numbers of the n-color palindrome compositions

$$cp_1(t) = Pb(t)\left[\chi(t) + \chi b(t)\right] = \frac{t^3 + 3t^2 + t}{t^4 - 3t^2 + 1}.$$

□

Example 5. *We make the patterns of n-color palindrome compositions of* 7 *and* 9*;*

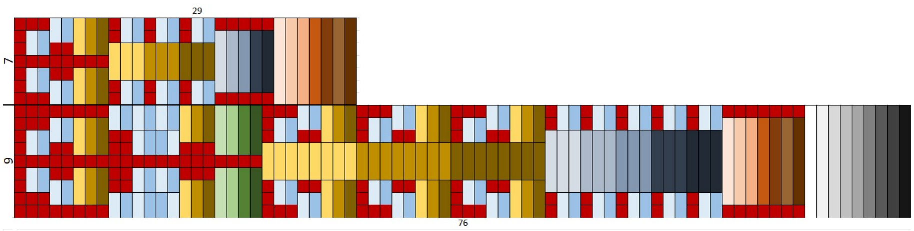

Figure 3: The patterns of n-color palindrome compositions of 7 and 9 such that the middle part is n-color

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An investigation on least square method in non-linear mathematical models

Mevlut Caylak *[1] *and Fusun Yalcin* [2]

In classical linear regression models, the parameters of the model seem to be linear. However, mathematical models that are used to explain daily life events are not easy to model linearly. Therefore, non-linear modeling might be required in such cases. On the other hand, at least one parameter of the model seems to be non-linear in non-linear regression models. While different methods are used in the prediction of these models, the least squares method is examined in this study.

2020 MSC: 62-06, 62G08, 93E24

KEYWORDS: Least squares, Statistics, Mathematical models

Introduction

In classical linear regression models, the parameters of the model seem to be linear. On the other hand, at least one parameter of the model seems to be non-linear in non-linear regression models. In our daily lives, all models cannot be linearly modeled, and non-linear mathematical models are used in such cases. The prediction and implementation of the non-linear models are more difficult compared to the linear models, and their results are obtained by iteration using different methods [2]. Therefore, non-linear regression models give a result slower than linear regression models. Another difficulty with non-linear regression models is that their data do not exactly fit the Normal Distribution. Instead, methods based on asymptotic theory or large sample theory are used. With the development of information technologies, software applications of statistical analysis methods have also been developed, which has facilitated solving equations. Partial derivatives are mostly calculated by statistical software by employing numerical methods. The numerical methods, which are used instead of analytical methods, usually produce approximate results. The negative exponential, Brody, Gompertz, Logistic, and Bertalanffy-Richards non-linear growth models are the most widely used non-linear modeling methods in the literature.

Materials and methods

Some of the models that seem to have non-linear mathematical structures cannot be linearized, no matter what transformation is performed. Since such models seem to be non-linear and we want to indicate that their actual structure is non-linear, they are called non-linear models. For example, the following logistic growth model is actually non-linear:

$$Y_i = \frac{\alpha}{1 + \beta e^{-kX_i}} + \varepsilon_i.$$

This model cannot be linearized whichever transformation is performed. The parameters of such models are estimated by using iterative methods.

The inability to linearize models that are not actually linear does not allow these models to be predicted like linear models. Various methods are used to predict these models, and almost all of them are iterative methods. The non-linear least squares method is one of these models.

Non-linear least squares method

In general, the model whose parameters are to be estimated can be expressed as follows:

$$Y_i = f(X_{j,}\theta) + \varepsilon_i$$

$$i = 1, 2, ..., n \text{ and } j = 2, 3, .., k.$$

In this equation, X denotes the vector of variables while θ denotes the vector of the parameters to be predicted. K denotes the number of parameters to be predicted. Let us suppose that the given function is non-linear. If the function were linear, the least squares method would be used to minimize it concerning θ by employing the following equation:

$$S = \sum_{i=1}^{n} [Y_i - E(Y_i)]^2.$$

Since the function does not need to be linear to apply the least squares method, non-linear functions can also be minimized by calculating partial derivatives with respect to the parameters. Thus, the S value can be minimized as follows:

$$S = \sum_{i=1}^{n} [Y_i - f(X_j, \theta)]^2.$$

The derivative of the function with respect to the parameters will be as follows:

$$\frac{\partial f(X_j, \theta)}{\partial \theta_j}.$$

Therefore, the normal equations can be generally obtained as follows:

$$\sum_{i=1}^{n} [Y_i - f(X_{j,}\theta)] \left[\frac{\partial f(X_i, \theta)}{\partial \theta_j} \right]_{\theta = \widehat{\theta}}.$$

The predictors of the parameters can be determined by using this general equation.

$$Y_i = \beta_0 - \beta_0 e^{-\beta_1 X_1} + \varepsilon_i.$$

The above equation is a non-linear model that cannot be linearized by transformation, and normal equations can be obtained by calculating partial derivatives as follows:

$$\sum_{i=1}^{n} [Y_i - \hat{\beta}_0 - \hat{\beta}_0 e^{-\hat{\beta}_1 X_1}][1 + e^{-\hat{\beta}_1 X_1}] = 0,$$

$$\sum_{i=1}^{n} [Y_i - \hat{\beta}_0 - \hat{\beta}_0 e^{-\hat{\beta}_1 X_1}][-\hat{\beta}_0 X_i e^{-\hat{\beta}_1 X_1}] = 0.$$

The challenges to be encountered in predicting parameters can be understood from the normal equations obtained. As it is clear, it is very difficult to predict $\hat{\beta}_0$ and $\hat{\beta}_1$. Therefore, it can be stated that the non-linear least squares method can only be used for simpler models that are not actually linear. Some software applications actually predict the coefficients of non-linear regression models using this method [3, 1].

Conclusion

The regression models, which are used in many fields to explain the events encountered in daily life, are widely used in the literature as well. This study presents an overview of the non-linear least squares method.

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Mathematical model selection in determining the course of infectious diseases

Melisa Gaygisiz *[1] *and Fusun Yalcin* [2]

From the past to the present, infectious diseases have negatively affected human life and even resulted in the death of many people. For this reason, estimating the rate of spread of any disease by scientists and commenting on its course have an important place in terms of precautions to be taken. In this context, the rate of spread of the disease was estimated with the help of the SI Model, using the HIV data we received from the Ministry of Health. The diffusion rate was found to be $\beta = 0,29$.

2020 MSC: 92-10, 65-06, 62-06, 34-06

KEYWORDS: HIV, Infectious diseases, Mathematical modeling

Introduction

Different mathematical models are used to study and analyze infectious diseases.

Kermack and McKendrick modeled the spread of infectious diseases in their study, assuming that not everyone is initially immune to a certain disease in a certain population, but they become immune to this disease after being infected. This model is known in the literature as the SIR model. While we were creating the model, ordinary differential equations were used. This model has greatly contributed to the development of mathematical epidemiology [3].

AlSaedi and Hameed modeled the Covid-19 pandemic in Iraq with the SIR mathematical model. In this study, infection, case fatality vs cure rate, reproductive number R_0 and growth factor of true accumulation infection showed significant similarities between reported data and generated estimate [1].

Ucakan analyzed the impact of tuberculosis disease on Turkey in her doctoral thesis using SIR, SEIR and BSEIR mathematical models. First, he looked at the stability analysis of the models, then applied the models to the data set obtained from the Ministry of Health. He referred to the effect of vaccination on the spread of the virus. He also emphasized the negative effects of migration to Turkey and the Covid-19 epidemic on the spread of tuberculosis in recent years. The importance of the measures to be taken for future epidemics was emphasized [4].

In addition, many models have been used so as to model diseases in many studies.

Materials and methods

Inspired by the SIR model used by the University of California in the spread of the Covid-19 epidemic, I made a simple analysis of the HIV data of 1985-2011 in Turkey with the help of the SI model.

SI mathematical model:

In this model, the population is divided into two groups as susceptible and infected. Individuals in the susceptible population get sick and remain ill for life. Individuals do not develop immunity. There is no incubation period in this model. HIV/AIDS is modeled using this model. In this model, there is a transition from the susceptible group to the infected group, but not from the infected group to the susceptible group. In other words, there is no improvement in the person. This model is

$$\frac{dS}{dt} = -\frac{\beta SI}{N},$$

$$\frac{dI}{dt} = \frac{\beta SI}{N}.$$

Here, susceptible population is represented with S, and infected population is represented with I. Conservation of population $S(t) + I(t) = N$, where N denotes the total population. So, N is a natural number. It is $S(0) + I(0) = N$ with $S(0) > 0, I(0) > 0$ initial conditions. Thus, and if we substitute $(N - I)$ for S, the time dependent change of the disease can be defined as shown below and t represents time in years;

$$\frac{dI}{dt} = \beta I \left(1 - \frac{1}{N}\right).$$

The relationship between population classes is demonstrated in the Figure 1 [4].

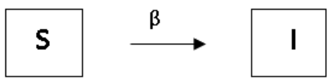

Figure 1: SI model chart

Result

Mathematical models have a very important place in understanding and interpreting epidemic studies. It may not always be possible to reach analytical solutions with the help of differential equations that make up the model. In such cases, numerical solution methods come into play and approximate solution of the system is obtained. In this study, HIV data obtained from the Ministry of Health's https://hsgm.saglik.gov.tr/tr/bulasici-hastaliklar/hiv-aids/hiv-aids-liste/hiv-aids-istatislik.html website was studied. The contact rate of HIV virus has been tried to be estimated approximately. For this, the data including the number of HIV patients taken by years was tried to be modeled with the help of the SI model. MATLAB R2020b was used for this estimate. Using the solution curve of the Covid-19 mathematical model published by UCI, the most appropriate β parameter was found and the contact rate of the virus was estimated. Birth and death parameters were not included in the model.

$$I(t) = \frac{NI_0}{I_0 + S_0 e^{-\beta t}}$$

$$I_0 = Infected(1)$$

$$S_0 = N - I_0$$

$$I = \frac{\beta N I_0}{I_0 + S_0 e^{-\beta t}} - Infected.$$

The sigmoid curve is approximated with the command "Estbeta = SIModel-Sigmoid()" to estimate the optimal β parameter. Since the population of 1985 is 50664458, N=50664458 is taken here. As a result of the approximation to the sigmoid curve, when Table 1 was examined, it was estimated as $\beta = 0.29485$.

Parameter	Meaning	Value
β	Contact Rate	0.29485

Table 1: β parameter value

This result shows us that the rate of spread of the disease is approximately 29% [2].

Conclusion

It was decided that SI mathematical modal is suitable modal for HIV data. With the help of the MATLAB R2020b program, the sigmoid curve published by the University of California (UCI) was approximated and the spread parameter of the disease was estimated as $\beta = 0,29485$. This has led us to the conclusion that healthy people have a probability of 0.29485 to pass into the sick class. It is aimed that the analyzes and statistical values obtained will contribute to the studies on the development of the model.

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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On the theory of zeta functions and their applications

Abdelmejid Bayad [1] *and Mounir Hajli* [*2]

The zeta functions play a fundamental role in number theory and mathematical physics. In this talk, we shall give an overview of the theory of abstract zeta functions, and outline some recent progress in the theory of Dirichlet series with applications to number theory, mathematical physics, and geometric analysis. This talk is based on our recent works [1, 2, 3, 4, 5].

2020 MSC: 11M35, 11F27, 11R59

KEYWORDS: Theta functions, Zeta functions, Special values

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Degenerate general bivariate Appell polynomials: Properties and application

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Determinantal representations of polynomials are important in many fields of mathematics, ranging from algebraic geometry to optimization. The motivation to introduce determinant expressions of special polynomials comes from the fact that they are useful in scientific computing in solving system of equations effectively. It is critical for this application to have determinantal representations not just for single valued polynomials but also for bivariate polynomials. In this article, a family of degenerate general bivariate Appell polynomials is introduced. Several different explicit representations, recurrence relations, and addition theorems are established for this family. With the aid of different recurrence relations, we establish the determinant expressions for the degenerate general bivariate Appell polynomials. We also establish determinant definitions for the degenerate general polynomials. Further, degenerate general bivariate Appell interpolation problem is considered. Several examples are framed as the applications of this family and their graphical representations are shown.

2020 MSC: 11B83, 11B68, 15A15, 33E99, 33C65

KEYWORDS: Degenerate general bivariate Appell polynomials, Determinant expressions, Degenerate bivariate Laguerre-Appell sequences, Interpolation hints

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Mathematical analysis for a ime delay model of Alzheimer's disease

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In this paper, we investigate a time-delayed model describing the evolution of Alzheimer disease (AD). We provide both necessary and sufficient conditions for the existence of steady states within this model. Next, we analyze the asymptotic behavior of the model and investigate the local asymptotic stability of each equilibrium point. Finally, we give some numerical simulation to illustrate our theoretical results.

2020 MSC: 92B05, 34K60,34K21

KEYWORDS: Alzheimer disease, Local stability, Constant delay

Introduction

Alzheimer disease (AD) is a neurodegenerative incurable disease of cerebral tissue that causes progressive and irreversible loss of mental functions such as memory, it's characterized by the presence of amyloide plaques. Amyloide plaques are small dense depots of a protein β-amyloide ($A\beta$) which is chemically adhesive and agglomerates progressively to form plaques.
There are a multitude paper that deals specifically with the modeling of the evolution of Alzheimer disease (AD), we can see [2], [4], [6] and [7].
Although factors that causes (AD) are still being investigated, recent studies such as [1] and [5] suggest that $A\beta$ oligomers (which are small aggregates of $A\beta$ monomers) after binding with healthy prions (PrPc) misfold these latter into a pathogenic form (PrPsc) that could be responsible for (AD).
In 2014, [6] proposed an in vivo model that takes into account for the first time prions in modeling the evolution of (AD). Their model consists of 4 species:

1. $A\beta$ oligomers concentration,
2. Prions PrPc concentration,
3. Concentration of complexes obtained by the biding of an oligomer and a prion,
4. The fourth equation describes the density of the insoluble plaque $A\beta$.

Description of the model

Our model is a system of three delayed differential equations, it's schematized in Figure 1.

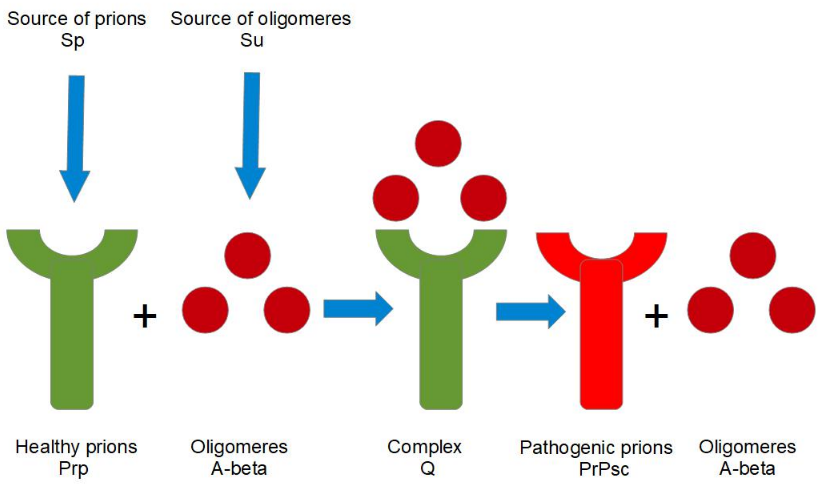

Figure 1: Representation of the model described by the system (1)

The first equation describes the temporal evolution of concentration of oligomers $A\beta$ denoted by the variable U.
In the second equation, concentration of healthy prions is denoted by P.
Finally, in the third equation the variable P_c represents the concentration of pathogenic prions.
We assume that there are fixed sources of oligomers and healthy prions denoted respectively S_u and S_p. Both oligomers, healthy and pathogenic prions are eliminated with a constant rates called respectively d_u, d_p and d_{pc}. Any prion can interact with $m \in \mathbb{N}^*$ oligomers to form a complex with a constant rate δ_1. Only after a certain time τ, the complex can split into the original m oligomers and a pathogenic prion with a constant rate δ_2. Under those assumptions, for $t \geq 0$, we obtain the following system:

$$\left\{ \begin{array}{lcl} \dot{U}(t) & = & S_u - d_u U(t) - \delta_1 m P(t) U^m(t) + m\delta_2 P(t-\tau) U^m(t-\tau) \\ \dot{P}(t) & = & S_p - d_p P(t) - \delta_1 P(t) U^m(t) \\ \dot{P}_c(t) & = & -d_{pc} P_c(t) + \delta_2 P(t-\tau) U^m(t-\tau). \end{array} \right. \quad (1)$$

For $t \in [-\tau, 0)$, the model is described with the same equations without delay, as prions and oligomers are not yet let out from the complex in the first τ units of time.
Table 1 summarize variables and parameters used in the system (1).

Table 1: Parameter description of the model

Parameter and variable	Definition	Unit
U	Concentration of oligomers $A\beta$	-
P	Concentration of healthy prions	-
P_c	Concentration of pathogenic prions	-
S_u	Source of oligomers $A\beta$	Days $^{-1}$
S_p	Source of healthy prions	Days $^{-1}$
d_u	Elimination rate of oligomers $A\beta$	Days $^{-1}$
d_p	Elimination rate of healthy prions	Days $^{-1}$
d_{pc}	Elimination rate of pathogenic prions	Days $^{-1}$
δ_1	Binding rate between oligomers and healthy prions	Days $^{-1}$
δ_2	Unbinding rate between oligomers and pathogenic prions	Days $^{-1}$
t	Time	Days
m	Number of oligomers in the complex	$\mathbb{N}^*$

Main results

Existence and non negativity of solutions

The following results are useful to prove global existence and non negativity of solutions, which can be found in the literature (see [8]). Consider the system of DDE:

$$\dot{x} = f(t, x(t), x(t-\tau)), \tag{2}$$

with a single delay $\tau > 0$. Let $s \in \mathbb{R}$ and $\phi : [s-r, s] \to \mathbb{R}^n$ be continuous. We seek a solution $x(t)$ of (2) that satisfies:

$$x(t) = \phi(t), \; s - \tau \leq t \leq s. \tag{3}$$

Theorem 1. *Let $f(t,x,y)$ and $f_x(t,x,y)$ be continuous on $\mathbb{R}^n$, $s \in \mathbb{R}$ and $\phi : [s-r, s] \to \mathbb{R}^n$ be continuous. Then there exists $\sigma > s$ and a unique solution of the initial value problem* (2)-(3) *on* $[s-r, \sigma]$.

Theorem 2. *Suppose that $f : \mathbb{R} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies the hypotheses of theorem 1 and*

$$\forall i, \quad t \quad \forall x, \quad y \in \mathbb{R}^n_+ : x_i = 0 \Rightarrow f_i(t,x,y) \geq 0. \tag{4}$$

If the initial condition $\phi \geq 0$, then the corresponding solution $x(t)$ of equation (2) satisfies $x(t) \geq 0$ for all $t \geq s$ where it is defined.

Proposition 3. *System* (1) *has a unique solution on* $[0, +\infty)$*. Furthermore, if initial conditions are non negative, solution remains non negative.*

In this section, we investigate existence and local stability of steady states.

Existence of steady states

As the following result suggests, the number of steady states depends on the parameter δ_2. Let

$$
\begin{cases}
G(U) &= \left(\frac{mS_p(\delta_2-\delta_1)}{(d_uU-S_u)} - \delta_1\right)U^m,\\
P_i^* &= \dfrac{S_p}{d_p+\delta_1 U_i^{*m}},\\
P_{ci}^* &= \dfrac{\delta_2 U_i^{*m}}{d_{pc}}\left(\dfrac{S_p}{d_p+\delta_1 U_i^{*m}}\right),\\
U_0 &= \dfrac{S_u}{d_u},\\
U_1 &= U_0 + \frac{(m-1)(\delta_2-\delta_1)S_p-\sqrt{\Delta_1}}{2\delta_1 d_u},\\
U_2 &= U_0 + \frac{(m-1)(\delta_2-\delta_1)S_p+\sqrt{\Delta_1}}{2\delta_1 d_u},\\
U_3 &= U_0 + \dfrac{mS_p(\delta_2-\delta_1)}{d_u\delta_1},\\
\Delta_1 &= (\delta_2-\delta_1)S_p\left[(m-1)^2S_p(\delta_2-\delta_1)-4S_u\delta_1\right].
\end{cases}
$$

Theorem 4. *Let* $\delta^* = \left(\frac{4S_u}{(m-1)^2S_p}+1\right)\delta_1$*, then system (1) has the following assumption*

1. *If* $\delta_2 \leq \delta_1$ *then system* (1) *has one nontrivial equilibrium* E_1 *with* $0 < U_1^* \leq U_0$.
2. *If* $\delta_1 < \delta_2 < \delta^*$ *then system* (1) *has one nontrivial equilibrium* E_1 *with* $U_0 < U_1^* < U_3$.
3. *If* $\delta_2 = \delta^*$ *and* $d_p < G(U_1)$ *then system* (1) *has one nontrivial equilibrium* E_1 *with* $U_1 < U_1^* < U_3$.
4. *If* $\delta_2 = \delta^*$ *and* $d_p = G(U_1)$ *then system* (1) *has one nontrivial equilibrium* E_1 *with* $U_1^* = U_1$.
5. *If* $\delta_2 = \delta^*$ *and* $d_p > G(U_1)$ *then system* (1) *has one nontrivial equilibrium* E_1 *with* $U_0 < U_1^* < U_1$.
6. *If* $\delta_2 > \delta^*$ *and* $d_p < G(U_1)$ *then system* (1) *has one nontrivial equilibrium* E_3 *with* $U_2 < U_3^* < U_3$.
7. *If* $\delta_2 > \delta^*$ *and* $d_p = G(U_1)$ *then system* (1) *has two nontrivial equilibria* E_1 *with* $U_1^* = U_1$ *and* E_3 *with* $U_2 < U_3^* < U_3$.
8. *If* $\delta_2 > \delta^*$ *and* $G(U_1) < d_p < G(U_2)$ *then system* (1) *has three nontrivial equilibria* E_1 *with* $U_0 < U_1^* < U_1$, E_2 *with* $U_1 < U_2^* < U_2$ *and* E_3 *with* $U_2 < U_3^* < U_3$.
9. *If* $\delta_2 > \delta^*$ *and* $d_p = G(U_2)$ *then system* (1) *has two nontrivial equilibria* E_3 *with* $U_3^* = U_2$ *and* E_1 *with* $U_0 < U_1^* < U_1$.
10. *If* $\delta_2 > \delta^*$ *and* $d_p > G(U_2)$ *then system* (1) *has one nontrivial equilibrium* E_1 *with* $U_0 < U_1^* < U_1$.

Results are summarized in Table 2.

Table 2: Existence of equilibria depending on the parameter δ_2

δ_2	**number of equilibria**
$0 < \delta_2 \leq \delta_1$	one nontrivial equilibrium E_1
$\delta_1 < \delta_2 < \delta^*$	one nontrivial equilibrium E_1
$\delta_2 = \delta^*$	one nontrivial equilibrium E_1
$\delta_2 > \delta^*$	one nontrivial equilibrium E_3 if $d_p < G(U_1)$
	two nontrivial equilibria E_1 and E_3 if $d_p = G(U_1)$
	three nontrivial equilibria E_1 , E_2 and E_3 if $G(U_1) < d_p < G(U_2)$
	two nontrivial equilibria E_1 and E_3 if $d_p = G(U_2)$
	one nontrivial equilibrium E_1 if $d_p > G(U_2)$

Stability of steady states

In this section, we investigate local stability of equilibrium points of system (1). The linearized system around equilibria is given by

$$\begin{bmatrix} \dot{U}(t) \\ \dot{P}(t) \end{bmatrix} = A \begin{bmatrix} U(t) \\ P(t) \end{bmatrix} + B \begin{bmatrix} U(t-r) \\ P(t-r) \end{bmatrix}, \tag{5}$$

where A and B are

$$A = \begin{bmatrix} -d_u - \delta_1 m^2 P^* U^{*m-1} & -\delta_1 m U^{*m} \\ -\delta_1 m P^* U^{*m-1} & -d_p - \delta_1 U^{*m} \end{bmatrix}, \qquad B = \begin{bmatrix} m^2 \delta_2 P^* U^{*m-1} & \delta_2 m U^{*m} \\ 0 & 0 \end{bmatrix}.$$

The characteristic equation of the system (1) for the nontrivial equilibrium point $E_i = (U_i^*, P_i^*, P_{ci}^*)$ is

$$P(\lambda) = \lambda^2 + a_i \lambda + c_i - (b_i \lambda + d_i) e^{-\lambda \tau} = 0, \tag{6}$$

where

$$\begin{aligned} a_i &= d_u + d_p + \delta_1 m^2 P_i^* U_i^{*m-1} + \delta_1 U_i^{*m}, \\ c_i &= d_p d_u + \delta_1 m^2 d_p P_i^* U_i^{*m-1} + \delta_1 d_u U_i^{*m}, \\ b_i &= \delta_2 m^2 P_i^* U_i^{*m-1} \text{ and} \\ d_i &= \delta_2 m^2 d_p P_i^* U_i^{*m-1}. \end{aligned}$$

Let

$$\begin{aligned} C_1 &= \frac{2\delta_1 S_u + (m-1)(\delta_2 - \delta_1) S_p - \sqrt{\Delta_1}}{2\delta_1}, \\ C_2 &= \delta_1 C_2^m \left(\frac{(m+1) S_p (\delta_2 - \delta_1) + \sqrt{\Delta_1}}{(m-1)(\delta_2 - \delta_1) S_p - \sqrt{\Delta_1}} \right), \\ C_3 &= \sqrt[m+1]{\frac{C_2}{\delta_1 C_1^m} [C_2 + \delta_1 C_1^m]}, \\ C_4 &= \frac{2\delta_1 S_u + (m-1)(\delta_2 - \delta_1) S_p + \sqrt{\Delta_1}}{2\delta_1}, \\ C_5 &= \delta_1 C_4^m \left(\frac{(m+1) S_p (\delta_2 - \delta_1) - \sqrt{\Delta_1}}{(m-1)(\delta_2 - \delta_1) S_p + \sqrt{\Delta_1}} \right) \text{ and} \\ C_6 &= \sqrt[m+1]{\frac{C_5}{\delta_1 C_4^m} [C_5 + \delta_1 C_4^m]}. \end{aligned}$$

Without delay ($\tau = 0$), we have the following result.

Theorem 5. *1. If $0 < \delta_2 \leq \delta_1$, then the unique equilibrium E_1 is asymptotically locally stable.*

2. *If $\delta_1 < \delta_2 < \delta^*$, then the unique equilibrium E_1 is asymptotically locally stable for $d_p \geq d_u$.*
3. *If $\delta_2 = \delta^*$ and $d_u \leq d_p < G(U_1)$, then the unique equilibrium E_1 is asymptotically locally stable.*
4. *If $\delta_2 = \delta^*$ and $d_p = G(U_1)$, then the unique equilibrium E_1 is unstable for $d_u > C_3$.*
5. *If $\delta_2 = \delta^*$ and $d_p > Max(d_u, G(U_1))$, then the unique equilibrium E_1 is asymptotically locally stable.*
6. *If $\delta_2 > \delta^*$ and $d_u \leq d_p < G(U_1)$, then the unique equilibrium E_3 is asymptotically locally stable.*
7. *If $\delta_2 > \delta^*$ and $d_p = G(U_1)$, then the equilibrium E_1 is unstable for $d_u > C_3$.*
8. *If $\delta_2 > \delta^*$ and $d_u \leq d_p = G(U_1)$, then the equilibrium E_3 is asymptotically locally stable.*
9. *If $\delta_2 > \delta^*$ and $Max(d_u, G(U_1)) < d_p < G(U_2)$, then the equilibrium E_1 is asymptotically locally stable.*
10. *If $\delta_2 > \delta^*$ and $G(U_1) < d_p < G(U_2)$, then the equilibrium E_2 is unstable.*
11. *If $\delta_2 > \delta^*$ and $Max(d_u, G(U_1)) < d_p < G(U_2)$, then the equilibrium E_3 is asymptotically locally stable.*
12. *If $\delta_2 > \delta^*$ and $d_u \leq d_p = G(U_2)$, then the equilibrium E_1 is asymptotically locally stable.*
13. *If $\delta_2 > \delta^*$ and $d_p = G(U_2)$, then the equilibrium E_3 is unstable for $d_u > C_6$.*
14. *If $\delta_2 > \delta^*$ and $d_p > Max(d_u, G(U_2))$, then the unique equilibrium E_1 is asymptotically locally stable.*

When $\tau > 0$, we have the following results.

Theorem 6. *1. If $0 < \delta_2 < \delta_1$, then the unique equilibrium E_1 is locally asymptotically stable for all $\tau > 0$.*

2. *If $\delta_2 = \delta^*$, $d_u \geq C_3$ and $d_p = G(U_1)$, the unique equilibrium E_1 is unstable for all $\tau > 0$.*
3. *If $\delta_2 > \delta^*$, $d_u \geq C_3$ and $d_p = G(U_1)$, then the equilibrium E_1 is unstable for all $\tau > 0$.*
4. *If $\delta_2 > \delta^*$ and $G(U_1) < d_p < G(U_2)$, then the equilibrium E_2 is unstable for all $\tau > 0$.*
5. *If $\delta_2 > \delta^*$, $d_u \geq C_6$ and $d_p = G(U_2)$, then the equilibrium E_3 is unstable for all $\tau > 0$.*

Numerical simulation

In this section, we give some numerical simulations for our model in order to illustrate our theoretical results (see subsection 2) and to see what happens in the non studied cases (see subsection 2). We discuss the simulation results of the system (1) according to the values of δ_2 and τ.

Case $\delta_2 \leq \delta^*$

Choosing $m = 3$, $d_u = 0.16$, $d_p = 0.13$, $d_q = 0.11$, $d_{pc} = 0.15$, $S_u = 0.5$ and $S_p = 0.6$, then the system will have a unique equilibrium E_1 which is locally asymptotically stable when $\tau = 0$ (see Figure 2).

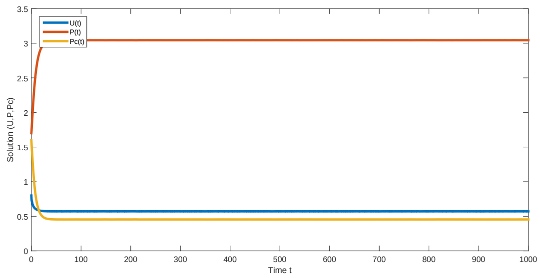

Figure 2: In this case, $\tau = 1.4$, $\delta_2 = 0.12 < \delta_1 = 0.36$, the unique equilibrium E_1 is locally asymptotically stable for initial condition $(0.8, 1.7, 1.6)$

Case $\delta_2 > \delta^*$ and $G(U_1) < d_p < G(U_2)$

Choosing $m = 3$, $\delta_1 = 0.36$, $d_{pc} = 0.15$, $S_u = 0.5$ and $S_p = 0.6$, then the system will have three equilibria: E_1, E_2 and E_3 (see Figures 3, 4 and 5).

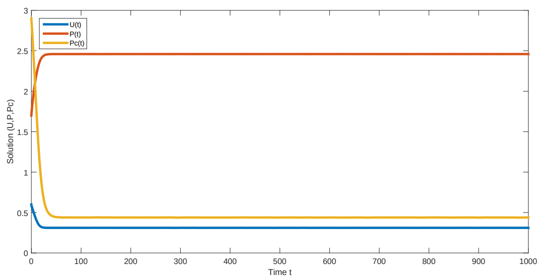

Figure 3: In this case, $\tau = 1.4$, $\delta_2 = 0.89 > \delta^* = 0.66$, $d_u = 1.98$ and $d_p = 0.23$, the equilibrium E_2 is unstable and it seems that E_1 is stable for initial condition $(0.6, 1.7, 2.9)$

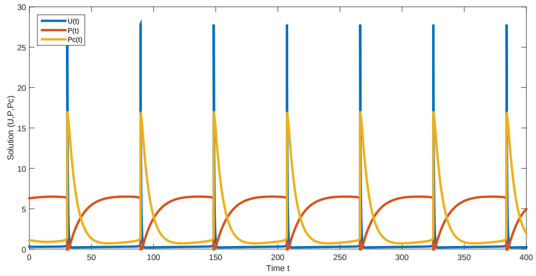

Figure 4: In this case, $\delta_2 = 0.89 > \delta^* = 0.66$, $d_p = 0.08$ and $d_u = 2.58$ with initial condition $(0.3, 6.3, 1.1)$, we observe the existence of oscillatory solutions for $\tau = 0$

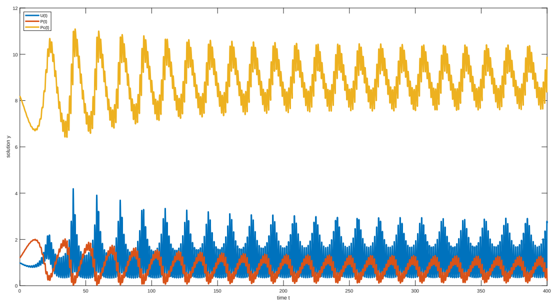

Figure 5: In this case, $\delta_2 = 0.89 > \delta^* = 0.66$, $d_p = 0.08$ and $d_u = 2.58$ with initial condition $(1, 1.2, 8.2)$, we observe the existence of oscillatory solutions for $\tau = 1.4$

Conclusion

In this work, we have developed a new mathematical model for Alzheimer disease, this model consists of a system of three delayed differential equations, it describes the dynamics of oligomers and prions. We have studied the existence of equilibria and their stability according to parameters of our model. In theorem (6), we have found conditions of stability and instability of equilibria for some values of parameters δ_1, δ_2, d_u and d_p. The remaining cases may need other methods to be analyzed. We have performed simulations for one instance where we can see oscillatory solutions, it will be interesting to investigate this case to find possible periodic solutions.

Acknowledgments

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Approximate solution to Riemann problem arising in hyperbolic conservation equation

Mahesh Kumar [*1] *and Ranjan Kumar Jana* [2]

In this work, we deliver an idea of understanding the advantage of the approximate analytical method namely homotopy analysis method (HAM) on hyperbolic conservation equations with specific piecewise initial value problems. Most of the hyperbolic conservation system solved by the numerical techniques due to possibility of discontinuity and sharp gradient in the solution. However, numerical methods require larger computational domain, discretization, round of error and sometimes round of error may cause a loss of accuracy. Hence in the present endeavor, it is natural to consider approximate analytical technique to the solution of nonlinear hyperbolic equations within less computational domain. To demonstrate the idea, we combine HAM with the method of characteristics approach and applied it to the hyperbolic conservation of inviscid Burgers' equation with specific Riemann problems. This study would in a way to demonstrate the potential and effectiveness of HAM to evaluate the various kinds of nonlinear equation arising in the gas dynamics theory.

2020 MSC: 35L02, 35L30, 41xx

Keywords: HAM, Riemann problem, Burgers'

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A hybrid genetic algorithm for the three-index assignment problem

Mohamed Mehbali

This paper addresses the axial three-index assignment problem (3IAP), also referred to as the axial multi-dimensional assignment problem. The problem is to allocate n jobs to n machines in n factories, with each job executed by one machine in a factory while minimizing the total cost. The combinatorial optimization problem 3IAP is considered an extension of the classical two-dimensional assignment problem and is proven to be NP-hard due to its inextricable nature.

We propose a hybrid genetic algorithm that combines a genetic algorithm with a local search method. Extensive experimentation demonstrates the efficiency of our approach in terms of solution quality and computational time. The results are compared with those obtained using the Branch-and-Bound (B&B) algorithm on medium to large 3IAP instances. The findings indicate that our hybrid genetic algorithm returns similar or better solutions in a competitive time.

2020 MSC: 90-08, 90C10, 90C27

KEYWORDS: Three-index assignment problem, Genetic algorithm, Local search procedure, Hungarian method

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On inference in generalized integer-valued $GARCHX$ models with structural changes

Mohamed Djemaa Sadoun *[1] *and Abderaouf Khalfi* [2]

We consider both estimation and testing problem in a generalized $INGARCH$ model with structural changes including exogenous covariates (referred to as $GCP-INGARCHX$). This class of models belongs to the observation-driven type models with regimes change where the regime-switching is driven by certain failure points occuring in the time. The conditional least squares (CLS) and the conditional maximum likelihood (CML) estimators of the underlying parameters are obtained for both cases that the break points are known or not. An off-line method based on a statistic of $CUSUM$ type is used to test the existence of change points. A simulation study and an application on real data set are provided to assess the performance of the model.

2020 MSC: 62F12, 62M10

KEYWORDS: Integer-valued process of counts, No-stationary $INGARCH$ model, (CLS) estimators, (CML) estimators, $CUSUM$-based test

Introduction and model formulation

In count time series analysis, Poisson distribution is frequently used and provides a classical framework (see Fokianos (2012)). However, due to the fact that many time series of counts encountered in practice exhibit the following features: the nonlinearity, the over and/or the under-dispersion, the multimodality, and the zeros-inflation, as well as the non-stationarity. Thus, a simple counts model with a regime-switching of change point-type including generalized and more flexible distributions is then needed to capture these characteristics often found in real-life examples. We propose a generalized integer-valued $GARCH$ model with structural changes containing exogenous covariates (hereafter referred to as $GCP-INGARCHX$). Our formulated model is a generalized $INGARCHX$ regime-switching model, where the regime-switching is driven by a certain latent random variable defined by pieces in time. Thus, we propose a more general change point $INGARCHX$ model, which includes several distributions for defining numerous $INGARCH$ model cases, and where "X" refers to specific exogenous covariates. Our modeling approach generalizes some famous works in the time series of counts modeling, namely: the model introduced by Ferland *et al* (2006), the model introduced by Wang *et al* (2014), the nonlinear proposed model by Chen *et al* (2019), and the generalized mixture $INGARCH$ model proposed by Mao et al (2019), and recently the Poisson and Negative Binomial $INGARCH$ models proposed by Lee *et al* (2020) in their change-point test for the conditional mean of time series of counts. In this contribution, we consider a simple version of the first-order Generalized Change-Point $INGARCHX$ model (hereafter referred to as $GCP-INGARCHX\,(1,1,q)$). The model is formulated by equation (1) below involving some observable count variable Z_t seen as a mixture of generalized

$INGARCHX$ components $Y_{k,t}$. Here the mixture is realized using a sequence of time-segmented variables.

$$\begin{cases} Z_t = \sum_{k=1}^{m+1} \mathbf{1}\left(S_t = k\right) Y_{S_t,t} \\ \mathbb{E}\left(Y_{S_t,t}|\mathcal{F}_{t-1}\right) = \lambda_{S_t,t} \\ var\left(Y_{S_t,t}|\mathcal{F}_{t-1}\right) = \upsilon_{S_t,0}\lambda_{S_t,t} + \upsilon_{S_t,1}\lambda_{S_t,t}^2 \\ \lambda_{S_t,t} = \alpha_{S_t,0} + \alpha_{S_t,1}\, Z_{t-1} + \beta_{S_t}\, \lambda_{S_t,t-1} + \sum_{l=1}^{q} \gamma_{S_t,l} X_{t-l} \end{cases} \tag{1}$$

where $\mathcal{F}_t$ indicates the information given up to time t. $\upsilon_{S_t,k} \geq 0, k = 0, 1$, but not simultaneously equal to zero, $\alpha_{S_t,0} > 0$, $\alpha_{S_t,i} \geq 0$, $\beta_{S_t,j} \geq 0$, and $\gamma_{S_t,l} \geq 0$ for $i = 1, ..., P$, $j = 1, ..., Q$, and $l - 1, ..., q$. $\mathbf{1}(.)$ denotes the indicator function. $\{S_t\}_{t\in\mathbb{Z}}$ is a sequence of independent and identically distributed random variables defined by pieces in time, which is defined as follows:

$$S_t = \begin{cases} 1, & if\ t \leq c_1 \\ 2, & if\ c_1 < t \leq c_2 \\ \vdots & \vdots \\ m & if\ c_{m-1} < t \leq c_m \\ m+1 & if\ t > c_m \end{cases}$$

where $(c_1, c_2, ..., c_m)'$ stand the vector of unknown break points. Furthermore, it is assumed that Z_{t-1} and S_t are independent. Note that $GCP - INGARCHX\,(1, 1, q)$ model could be extended to $GCP - INGARCHX\,(P, Q, q)$ model, where the change occurs only on the link function $\lambda_{S_t,t}$ by increasing the order of lags as follows:

$$\lambda_{S_t,t} = \alpha_{S_t,0} + \sum_{i=1}^{P} \alpha_{S_t,i}\, Z_{t-i} + \sum_{j=1}^{Q} \beta_{S_t,j}\, \lambda_{S_t,t-j} + \sum_{l=1}^{q} \gamma_{S_t,l} X_{t-l}. \tag{2}$$

Where $\alpha_{S_t,0} > 0$, $\alpha_{S_t,i} \geq 0$, $\beta_{S_t,j} \geq 0$, and $\gamma_{S_t,l} \geq 0$ for $i = 1, ..., P$, $j = 1, ..., Q$, and $l = 1, ..., q$. While supposing that Z_{t-i} and S_t are independent for all t and $i > 0$.

Main result

We consider the problem of testing whether or not a change has occurred in $GINGARCH\,(1, 1)$ model. Let $Z_1, \ldots, Z_n$ be consecutive observations from the model (1). We propose a change point test (only one change occurred) based on the $CUSUM$ statistic. We look for a test solving the following decision problem:

$$H_0 : c_1 = n \text{ versus } H_1 : c_1 < n$$
$$H_0 : \theta_1 = \theta_2 = \theta \text{ versus } H_1 : \theta_1 \neq \theta_2.$$

We use the maximum of the normalized $CUSUM$ statistics:

$$T_n = \max_{1\leq k\leq n} \sqrt{\frac{n}{k\,(n-k)}}\widehat{S}_n\,(c_1)\,.$$

We can stand on two approaches based $CUSUM$ test, namely: Estimates-based $CUSUM$ test ($CLSE-$based CUSUM test (Kang *et al* (2009)), and $CMLE-$based CUSUM test (Kang *et al* (2014)). Residual-based $CUSUM$ test (more stable test, Franke (2012)). We shall adapt the works of Kang *et al* (2014).

Theorem 2.1. *Suppose that some milder conditions are hold. We have*:

$$T_n = \max_{1\leq k\leq n} \frac{1}{\sqrt{n}\widehat{\sigma}_n} \left| \sum_{t=1}^{k} \widehat{\varepsilon}_t - \left(\frac{k}{n}\right) \sum_{t=1}^{n} \widehat{\varepsilon}_t \right|,$$

where $\widehat{\sigma}_n = \dfrac{\sum_{t=1}^{n} \widehat{\varepsilon}_t^2}{n}$. *Then under* H_0 $T_n \rightsquigarrow \text{supp}_{0\leq s\leq 1}\mathbb{B}_1(\text{s})$, *where* $\mathbb{B}_1(s)$ *is one dimensional Brownian bridge, and we reject* H_0 *if* T_n *is large.*

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Investigation of copper filled carbon nanotubes

Mansoor H. Alshehri

Carbon nanoubes have attracted tremendous interest in the research community and starting-point for the development of nanotechnology. In this study, classical applied mathematical modeling is used to investigate introducing copper (Cu) atoms inside different types of single-walled carbon nanotubes (SWCNTs), as shown in Fig. 1. The explicit analytical expressions are derived to study the encapsulation for both zigzag and armchair carbon nanotubes. The Lennard-Jones potential function together with the continuous approach are adopted to obtain minimum interaction energies of this system to determine the preferred radius size of the tube to enclose the copper atoms. The results show that the optimal size of SWCNT to encapsulate a Cu atom is ≈ 3.35 Å, so it can be inferred that the armchair $(5, 5)$ and zigzag $(10, 0)$ tubes are the preferred tubes.

2020 MSC: 65Z05, 53Z05

Keywords: Copper, Nanotube, Mathematical modeling, Continuous approach, Lennard-Jones potential

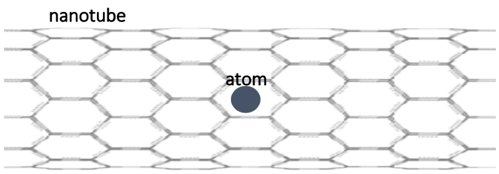

Figure 1: Geometrical diagram of an atom in nanotube

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Generalizations of Euler-Grüss type inequalities

Mihaela Ribičić Penava

The main aim of this note is to present weighted generalizations of Euler-Grüss type inequalities by using the weighted integral formulae given in [3]. Also, we derive some improvements of previously obtained results. As applications of the main results, we obtain error estimates for the Gauss–Chebyshev quadrature rules.

2020 MSC: 25D15, 65D30, 65D32

Keywords: Euler type identity, Weighted integral formulae, Grüss inequality, Error estimate

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Identities on Simsek numbers: Approach to generating functions with Faà di Bruno's formula

Neslihan Kilar

The aim of this paper is to give some formulas for the Peters-type Simsek numbers of the first kind by virtue of Faà di Bruno's formula. By using this formula and generating function methods, we obtain novel identities, which are combinatorial sums, associated with the Peters-type Simsek numbers of the first kind, the Stirling numbers of the first kind, and the Stirling numbers of the second kind.

2020 MSC: 05A19, 11B73, 11B83

KEYWORDS: Faà di Bruno formula, Stirling numbers, Bell polynomials, Peters-type Simsek numbers of the first kind

Introduction

It is an old problem to find an explicit expression for the n-th derivative of composition function. In the literature, there are many ways to represent the n-th derivative of composition function. One of them is Faà di Bruno's formula. This formula has many applications in areas of mathematics, such as real analysis, combinatorics analysis, partition theory, mathematical statistics, matrix theory, and others (see for details [3]-[7], [12, 19]). In this paper, we investigate the Peters-type Simsek numbers of the first kind with the help of Faà di Bruno's formula and generating function methods. We give some formulas, including these numbers and the Stirling numbers of the first and second kinds.

Throughout of this paper, we use the following notations and definitions:

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, real numbers, and complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Moreover, the falling factorial $(x)_m$ and the rising factorial $x^{(m)}$ are defined by

$$(x)_m = x\,(x-1)\,(x-2)\ldots(x-m+1)$$

and

$$x^{(m)} = x\,(x+1)\,(x+2)\ldots(x+m-1)$$

with $(x)_0 = x^{(0)} = 1$ (*cf.* [1]-[19]).

The Stirling numbers of the first kind are defined by

$$(x)_m = \sum_{k=0}^{m} S_1(m,k)\,x^k$$

and their generating function is given as follows:

$$\frac{(\ln(1+t))^k}{k!} = \sum_{m=0}^{\infty} S_1(m,k)\,\frac{t^m}{m!}$$

(*cf.* [1]-[4], [8]-[19]).

The Stirling numbers of the first kind satisfy the following recurrence relation:

$$S_1(m+1,k) = S_1(m,k-1) - mS_1(m,k), \tag{1}$$

where $0 < k < m$, with

$$S_1(m,m) = 1 \quad (m \in \mathbb{N}_0); \quad S_1(m,0) = S_1(0,m) = 0 \quad (m \in \mathbb{N})$$

and for $k > m$, $S_1(m,k) = 0$ (*cf.* [1]-[4], [8]-[19]).

By using (1), the numbers $S_1(m,k)$ are given in Table 1:

$m \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	-1	1			
3	0	2	-3	1		
4	0	-6	11	-6	1	
5	0	24	-50	35	-10	1

Table 1: Some values of the Stirling numbers of the first kind

The Stirling numbers of the second kind are defined by

$$x^m = \sum_{k=0}^{m} S_2(m,k)(x)_k$$

and their generating function is given as follows:

$$\frac{(e^t-1)^k}{k!} = \sum_{m=0}^{\infty} S_2(m,k) \frac{t^m}{m!}$$

(*cf.* [1]-[4], [8]-[19]).

The Stirling numbers of the second kind satisfy the following recurrence relation:

$$S_2(m+1,k) = kS_2(m,k) + S_2(m,k-1), \tag{2}$$

where $0 < k < m$, with

$$S_2(m,m) = 1 \quad (m \in \mathbb{N}_0); \quad S_2(m,0) = S_2(0,m) = 0 \quad (m \in \mathbb{N})$$

and for $k > m$, $S_2(m,k) = 0$ (*cf.* [1]-[4], [8]-[19]).

By using (2), the numbers $S_2(m,k)$ are given in Table 2:

$m \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

Table 2: Some values of the Stirling numbers of the second kind

The Peters-type Simsek numbers of the first kind, $Y_m(\lambda)$, are defined by

$$\frac{2}{\lambda^2 t+\lambda-1}=\sum_{m=0}^{\infty} Y_m(\lambda)\frac{t^m}{m!}, \tag{3}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) (*cf.* [14, Eq. (2.13)]).

Note that many different types of Simsek numbers and polynomials have been defined in recent years. These numbers and polynomials are members of the family of the Peters numbers and polynomials, and also the Boole numbers and polynomials. Here, we will deal with the Peters-type Simsek numbers of the first kind. These numbers have applications in many branches of mathematics, not only in combinatorics and probability distribution functions but also in algebra. It should also be noted that these numbers have been studied by many authors (*cf.* [8]-[11], [16]-[18]).

The Peters-type Simsek numbers of the first kind $Y_m(\lambda)$ are computed by the following formula:

$$Y_m(\lambda)=2(-1)^m\frac{m!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^m, \tag{4}$$

where $m \in \mathbb{N}_0$ (*cf.* [14, Eq. (2.18)]).

By using (4), we compute a few values of the numbers $Y_m(\lambda)$ as follows:

$$\begin{aligned}
Y_0(\lambda) &= \frac{2}{\lambda-1}, \quad Y_1(\lambda)=-\frac{2\lambda^2}{(\lambda-1)^2},\\
Y_2(\lambda) &= \frac{4\lambda^4}{(\lambda-1)^3}, \quad Y_3(\lambda)=-\frac{12\lambda^6}{(\lambda-1)^4},\\
Y_4(\lambda) &= \frac{48\lambda^8}{(\lambda-1)^5}, \quad Y_5(\lambda)=-\frac{240\lambda^{10}}{(\lambda-1)^6},
\end{aligned}$$

and so on.

The numbers $Y_m^{(k)}(\lambda)$ are defined by

$$\left(\frac{2}{\lambda^2 t+\lambda-1}\right)^k=\sum_{m=0}^{\infty} Y_m^{(k)}(\lambda)\frac{t^m}{m!}, \tag{5}$$

where $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) (*cf.* [10, Eq. (2.1)]).

When $k=1$ in (5), we have

$$Y_m(\lambda)=Y_m^{(1)}(\lambda).$$

By using (5), the numbers $Y_m^{(k)}(\lambda)$ satisfy the following relation:

$$Y_{m+v}^{(k)}(\lambda)=2^{-v}(-1)^v\lambda^{2v}k^{(v)}Y_m^{(k+v)}(\lambda) \tag{6}$$

(*cf.* [10, Eq. (4.1)]).

The exponential partial Bell polynomials, $B_{m,k}(x_1,x_2,\ldots,x_{m-k+1})$, are defined by means of the following generating function:

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} x_j\frac{t^j}{j!}\right)^k=\sum_{m=k}^{\infty} B_{m,k}(x_1,x_2,\ldots,x_{m-k+1})\frac{t^m}{m!},$$

where $k \in \mathbb{N}_0$ (*cf.* [2, 3, 12, 13]).

The exponential partial Bell polynomials are also defined by

$$B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \sum \frac{m!}{j_1! j_2! \cdots j_{m-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \times \cdots \left(\frac{x_{m-k+1}}{(m-k+1)!}\right)^{j_{m-k+1}}, \tag{7}$$

where the sum is taken over all sequences $j_1, j_2, \ldots, j_{m-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \cdots + j_{m-k+1} = k$$

and

$$j_1 + 2j_2 + 3j_3 + \cdots + (m-k+1)\, j_{m-k+1} = m$$

(*cf.* [2, 3, 12, 13]).

From (7), the polynomials $B_{m,k}(x_1, x_2, ..., x_{m-k+1})$ are given in Table 3:

$m\backslash k$	0	1	2	3	4	5
0	1					
1	0	x_1				
2	0	x_2	x_1^2			
3	0	x_3	$3x_1x_2$	x_1^3		
4	0	x_4	$3x_2^2 + 4x_1x_3$	$6x_1^2x_2$	x_1^4	
5	0	x_5	$10x_2x_3 + 5x_1x_4$	$15x_1x_2^2 + 10x_1^2x_3$	$10x_1^3x_2$	x_1^5

Table 3: Some values of the exponential partial Bell polynomials

The polynomials $B_{m,k}(x_1, x_2, \ldots, x_{m-k+1})$ are also related to the Stirling numbers of the first and second kinds, which are given as follows:

$$B_{m,k}\left(0!, -1!, 2!, \ldots, (-1)^{m-k}(m-k)!\right) = S_1(m,k)$$

and

$$B_{m,k}\left(\underbrace{1, 1, 1, \ldots, 1}_{m-k+1}\right) = S_2(m,k) \tag{8}$$

(*cf.* [2, 3, 7, 12, 13, 19]).

Let f and g be functions with a sufficient number of derivatives. Faà di Bruno's formula is given by

$$\frac{d^n}{dt^n} f(g(t)) = \sum \frac{n!}{k_1! k_2! \cdots k_n!} f^{(k)}(g(t)) \left(\frac{g'(t)}{1!}\right)^{k_1} \left(\frac{g''(t)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(n)}(t)}{n!}\right)^{k_n},$$

where $k = k_1 + k_2 + \cdots + k_n$ and the sum is over all different solutions in nonnegative integers of $k_1 + 2k_2 + \cdots + nk_n = n$ (*cf.* [3, 12, 13]).

Here we note that there are many versions of Faà di Bruno's formula. Some of these are given as follows: set partition version, Bell polynomial version, determinant version, Hoppe's formula, and others.

Faà di Bruno's formula is also defined in terms of the Bell polynomials as follows:

$$\frac{d^n}{dt^n} f(g(t)) = \sum_{k=1}^{n} f^{(k)}(g(t))\, B_{n,k}\left(g'(t), g''(t), \ldots, g^{(n-k+1)}(t)\right) \tag{9}$$

(*cf.* [2, 3, 12, 13]). The Bell polynomials version of Faà di Bruno's formula is also known as the Riordan's formula.

Main results

In this section, by using similar method in the work of Xu [19], which is Faà di Bruno's formula, we give some combinatorial identities including the Peters-type Simsek numbers of the first kind and some special numbers.

Theorem 1. *Let $n \in \mathbb{N}$. Then we have*

$$Y_n(\lambda) = 2\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{(-1)^j j!\lambda^{2j}}{(\lambda-1)^{j+1}}S_1(k,j)\,S_2(n,k). \tag{10}$$

Proof. Let

$$f(t) = \frac{2}{\lambda^2\ln(t)+\lambda-1} \quad \text{and} \quad g(t) = e^t.$$

By using Faà di Bruno's formula given in Eq. (9) with the above functions, we have

$$\frac{d^n}{dt^n}\left(\frac{2}{\lambda^2 t+\lambda-1}\right) = \sum_{k=1}^{n} f^{(k)}(g(t))\,B_{n,k}\left(e^t, e^t, \ldots, e^t\right).$$

From (3) and

$$Y_n(\lambda) = \frac{d^n}{dt^n}\left(\frac{2}{\lambda^2 t+\lambda-1}\right)|_{t=0},$$

the above equation reduces to the following result:

$$Y_n(\lambda) = \sum_{k=1}^{n} f^{(k)}(g(t))\,|_{t=0}\; B_{n,k}(1,1,\ldots,1).$$

Combining the above equation with (8), we obtain

$$Y_n(\lambda) = \sum_{k=1}^{n} f^{(k)}(g(t))\,|_{t=0}\; S_2(n,k). \tag{11}$$

Now, we find the expression $f^{(k)}(g(t))$. Namely,

$$\begin{aligned}
f'(g(t)) &= -\frac{2e^{-t}\lambda^2}{(\lambda^2 t+\lambda-1)^2},\\
f''(g(t)) &= 2e^{-2t}\left(\frac{\lambda^2}{(\lambda^2 t+\lambda-1)^2}+\frac{2\lambda^4}{(\lambda^2 t+\lambda-1)^3}\right),\\
f'''(g(t)) &= -2e^{-3t}\left(\frac{2\lambda^2}{(\lambda^2 t+\lambda-1)^2}+\frac{6\lambda^4}{(\lambda^2 t+\lambda-1)^3}+\frac{6\lambda^6}{(\lambda^2 t+\lambda-1)^4}\right),\\
f^{(4)}(g(t)) &= 2e^{-4t}\left(\frac{6\lambda^2}{(\lambda^2 t+\lambda-1)^2}+\frac{22\lambda^4}{(\lambda^2 t+\lambda-1)^3}+\frac{36\lambda^6}{(\lambda^2 t+\lambda-1)^4}\right)\\
&\quad +2e^{-4t}\left(\frac{24\lambda^8}{(\lambda^2 t+\lambda-1)^5}\right),\\
f^{(5)}(g(t)) &= -2e^{-5t}\left(\frac{24\lambda^2}{(\lambda^2 t+\lambda-1)^2}+\frac{100\lambda^4}{(\lambda^2 t+\lambda-1)^3}+\frac{210\lambda^6}{(\lambda^2 t+\lambda-1)^4}\right)\\
&\quad -2e^{-5t}\left(\frac{240\lambda^8}{(\lambda^2 t+\lambda-1)^5}+\frac{120\lambda^{10}}{(\lambda^2 t+\lambda-1)^6}\right),
\end{aligned}$$

and so on. Thus, for $k \in \mathbb{N}$, we obtain the following sum:

$$f^{(k)}\left(g\left(t\right)\right) = 2e^{-kt}\sum_{j=1}^{k}\frac{\left(-1\right)^{j} j!\lambda^{2j}}{\left(\lambda^{2}t+\lambda-1\right)^{j+1}}S_{1}\left(k,j\right).$$

Substituting $t=0$ into the above derivative equation, we have

$$f^{(k)}\left(g\left(t\right)\right)|_{t=0} = \ 2\sum_{j=1}^{k}\frac{\left(-1\right)^{j} j!\lambda^{2j}}{\left(\lambda-1\right)^{j+1}}S_{1}\left(k,j\right). \tag{12}$$

Combining (11) with (12), we have the following formula:

$$Y_{n}\left(\lambda\right) = 2\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{\left(-1\right)^{j} j!\lambda^{2j}}{\left(\lambda-1\right)^{j+1}}S_{1}\left(k,j\right)S_{2}\left(n,k\right).$$

Hence, the proof of theorem is complete. □

Substituting $k=1$ into (6), we get

$$Y_{n+v}\left(\lambda\right) = 2^{-v}(-1)^{v}\lambda^{2v}v!Y_{n}^{(v+1)}\left(\lambda\right). \tag{13}$$

Combining (13) with (10), after some calculations, we have the following corollary:

Corollary 2. *Let $n, v \in \mathbb{N}$. Then we have*

$$Y_{n}^{(v)}\left(\lambda\right) = \frac{(-1)^{v-1}2^{v}}{\lambda^{2v-2}(v-1)!}\sum_{k=1}^{n+v-1}\sum_{j=1}^{k}\frac{\left(-1\right)^{j} j!\lambda^{2j}S_{1}\left(k,j\right)S_{2}\left(n+v-1,k\right)}{\left(\lambda-1\right)^{j+1}}.$$

Conclusion

In this paper, by using the generating functions of some special numbers with Faà di Bruno's formula, we obtained some identities, including the Peters-type Simsek numbers of the first kind, the Stirling numbers of the first kind, and the Stirling numbers of the second kind. These results have the potential to be used by researchers in related areas.

In the near future, with the help of the results given in this paper and Faà di Bruno's formula, we will study and investigate certain families of special numbers and polynomials and their generating functions.

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Observations on asymptotic expressions for combinatorial Simsek numbers

Neslihan Kilar

In this paper, we investigate the combinatorial Simsek numbers of the sixth kind and their generating functions with asymptotic expressions. With the explicit formula of these numbers, we give some formulas for these numbers, including the Catalan numbers and the Stirling numbers. We also give some inequalities for the combinatorial Simsek numbers of the sixth kind. Finally, we present some observations on these numbers.

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Introduction

It is well known that in recent years, finite sums and binomial coefficients have been widely used in many fields, especially mathematics, physics, and other applied sciences. These sums and their coefficients are related both to some special families of numbers and polynomials and to Stirling's approximation. These relationships have been given by many authors (see for detail [5, 9, 13]). Thus, in this paper, we investigate both the asymptotic expressions of the combinatorial Simsek numbers of the sixth kind and some inequalities for these numbers.

We use the following notations and notations throughout this paper:

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of positive integers, real numbers, complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Stirling numbers of the second kind are defined by

$$\frac{(e^z-1)^r}{r!} = \sum_{m=0}^{\infty} S_2(m,r)\frac{z^m}{m!}, \tag{1}$$

where $r \in \mathbb{N}_0$ (*cf.* [1]-[16]).

The Catalan numbers are defined by

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{m=0}^{\infty} C_m z^m,$$

where $0 < |z| \leq \frac{1}{4}$, and

$$C_m = \frac{1}{m+1}\binom{2m}{m}, \tag{2}$$

where $m \in \mathbb{N}_0$ (*cf.* [6]).

The combinatorial Simsek numbers of the sixth kind, $y_6(m,r;\vartheta,p)$, are defined by

$$\frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}^p \vartheta^k e^{zk} = \sum_{m=0}^{\infty} y_6(m,r;\vartheta,p)\frac{z^m}{m!}, \tag{3}$$

where $r, p \in \mathbb{N}$ and $\vartheta \in \mathbb{R}$ (or $\mathbb{C}$) (*cf.* [9]).

The explicit formula for the combinatorial Simsek numbers of the sixth kind, which includes the higher order binomial coefficients, is as follows:

$$y_6\left(m, r; \vartheta, p\right) = \frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}^{p}\vartheta^{k}k^{m}, \tag{4}$$

where $m, r, p \in \mathbb{N}_0$ (*cf.* [9, Eq. (14)]). By using the same method that of [5] and with the help of Eq. (4), the following relations among the numbers $y_6\left(m, r; \vartheta, p\right)$, some special numbers and finite sums are given.

Putting $p = 1$ and $\vartheta = -1$ in (4), we have

$$y_6\left(m, r; -1, 1\right) = \frac{1}{r!}\sum_{k=0}^{r}(-1)^{k}\binom{r}{k}k^{m} = (-1)^{r}S_2\left(m, r\right), \tag{5}$$

putting $p = 2$, $\vartheta = 1$ and $m = 0$ in (4), we obtain

$$y_6\left(0, r; 1, 2\right) = \frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}^{2} = \frac{(r+1)C_r}{r!}, \tag{6}$$

putting $p = m = 2$ and $\vartheta = 1$ in (4), we get

$$y_6\left(2, r; 1, 2\right) = \frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}^{2}k^{2} = \frac{r^{3}C_{r-1}}{r!} \tag{7}$$

and putting $p = 2$, $\vartheta = 1$ and $m = 3$ in (4), we get

$$y_6\left(3, r; 1, 2\right) = \frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}^{2}k^{3} = \frac{r^{3}(r+1)C_{r-1}}{2r!} \tag{8}$$

(*cf.* [5, 9]).

Substituting $p = 1$ into (4), we also get

$$y_6\left(m, r; \vartheta, 1\right) = \frac{1}{r!}\sum_{j=0}^{r}\binom{r}{j}j^{m}\vartheta^{j} = y_1\left(m, r; \vartheta\right), \tag{9}$$

where $y_1\left(m, r; \vartheta\right)$ denotes the combinatorial Simsek numbers of the first kind and their generating function:

$$\frac{\left(\vartheta e^{z}+1\right)^{r}}{r!} = \sum_{m=0}^{\infty}y_1\left(m, r; \vartheta\right)\frac{z^{m}}{m!}, \tag{10}$$

where $r \in \mathbb{N}_0$, $\vartheta \in \mathbb{C}$ (*cf.* [10]).

Moreover, in [5], the author studied on the numbers $y_6\left(m, r; \vartheta, p\right)$. The author gave both asymptotic expressions for some special values of these numbers and many relations. The asymptotic expressions of the numbers $y_6\left(m, r; \vartheta, p\right)$ is given as follows:

$$y_6\left(0, r; 1, p\right) \sim \frac{2^{pr-1}}{\sqrt{p}}\left(\frac{e}{r}\right)^{r}\left(\frac{2}{\pi r}\right)^{\frac{p}{2}}, \tag{11}$$

where $r, p \in \mathbb{N}$ (*cf.* [5, Theorem 3.1.]).

In [5], the author also gave the following inequalities for the numbers $y_6(m, r; \vartheta, p)$:

$$\frac{2^r}{r!} \leq y_6(0, r; 1, 2) \leq \frac{2^r r^r}{(r!)^2} \tag{12}$$

and

$$\frac{2^{2r-1}}{r!\sqrt{r}} \leq y_6(0, r; 1, 2) \leq \frac{2^{2r}}{r!}, \tag{13}$$

where $r \in \mathbb{N}$.

Main results

In this section, by using Stirling's approximation, we give some asymptotic expressions for the combinatorial Simsek numbers of the sixth kind, $y_6(m, r; \vartheta, p)$. Further, we present lower and upper bounds for these numbers.

Combining Stirling's approximation

$$r! \sim \sqrt{2\pi r}\left(\frac{r}{e}\right)^r \tag{14}$$

(*cf.* [2, 15]) with (2), we have the following well-known result:

$$C_r \quad \sim \quad \frac{2^{2r}}{(r+1)\sqrt{\pi r}} \tag{15}$$

or

$$C_r \quad \sim \quad \frac{2^{2r}}{r\sqrt{\pi r}}$$

(*cf.* [6, P. 110]).

Combining (6) with (14) and (15), we have the following result:

Corollary 1 (*cf.* [5, Eq. (3.10)]). *Let $r \in \mathbb{N}$. Then we have*

$$y_6(0, r; 1, 2) \sim \frac{2^{2r-\frac{1}{2}} e^r}{\pi r^{r+1}}.$$

Combining (7) with (14) and (15), we get the following result:

Corollary 2. *Let $r \in \mathbb{N}$ with $r > 1$. Then we have*

$$y_6(2, r; 1, 2) \sim \frac{e^r 2^{2r-\frac{5}{2}}}{\pi r^{r-\frac{3}{2}}(r-1)^{\frac{1}{2}}}.$$

Combining (8) with (14) and (15), we obtain the following result:

Corollary 3. *Let $r \in \mathbb{N}$ with $r > 1$. Then we have*

$$y_6(3, r; 1, 2) \sim \frac{e^r (r+1) 2^{2r-\frac{7}{2}}}{\pi r^{r-\frac{3}{2}}(r-1)^{\frac{1}{2}}}.$$

By using the following lower and upper bounds for the Stirling numbers of the second kind:

$$\frac{1}{2}\left(r^2 + r + 2\right) r^{m-r-1} - 1 \leq S_2(m, r) \leq \frac{1}{2}\binom{m}{r} r^{m-r},$$

where $m \geq 2$ and $1 \leq r \leq m-1$ (*cf.* [8, 14]), and (5), we derive the following theorem:

Theorem 4. *Let $m, r \in \mathbb{N}$ with $m \geq 2$ and $1 \leq r \leq m-1$. Then we have*

$$\frac{1}{2}\left(r^2 + r + 2\right) r^{m-r-1} - 1 \leq (-1)^r y_6(m, r; -1, 1) \leq \frac{1}{2}\binom{m}{r} r^{m-r}.$$

Conclusion

In this paper, with the aid of Stirling's approximation, we obtained some asymptotic expressions for the combinatorial Simsek numbers of the sixth kind. We also obtained lower and upper bounds for the combinatorial Simsek numbers of the sixth kind.

Our future plan is not only to investigate the asymptotic expressions in more general cases of the combinatorial Simsek numbers of the sixth kind, which are include higher powers of binomial coefficients, but also to investigate other relations involving these numbers.

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Convolution sums involving restricted divisor functions for coprime conditions

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For a positive integer N, let $E(N) := \sum_{\substack{d|N \\ d\equiv 1 \pmod 3}} 1 - \sum_{\substack{d|N \\ d\equiv -1 \pmod 3}} 1$ and $\lambda(N) := E(N) - 3E(\frac{N}{3})$. The formula $\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t)$ for convolution sums is very well known. In this article, $\sum_{\substack{t=1 \\ \gcd(t,N-t)=1}}^{N-1} \lambda(t)\lambda(N-t)$ is calculated using arithmetical inverse of λ defined by Dirichlet convolution.

Keywords: Dirichlet convolution, Restricted divisor functions

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Taylor interpolation formula and Jensen-type inequalities' improvements

Neda Lovričević *[1], *Marija Bošnjak* [2], *Mario Krnić* [3] *and Josip Pečarić* [4]

General improvements of the Jensen inequality as well as of some related inequalities are derived by means of the Taylor interpolation formula. Moreover, compared with the existing ones, more accurate superadditivity and monotonicity relations for the Jensen functional are obtained in the process.

2020 MSC: 26D10, 26D15, 26A51

KEYWORDS: Jensen inequality, Taylor polynomial, Convexity

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Effects of vector control algorithms on motor types

Selma Nilay Savkliyildiz [*1] *and Yakup Irim* [2]

This study investigates the effects of vector control algorithms on different types of electric motors, including asynchronous motors and permanent magnet synchronous motors. Through mathematical modelling, various control techniques like Field-Oriented Control (FOC) and Direct Torque Control (DTC) are discussed in terms of efficiency, torque ripple, current ripple, and speed control accuracy.

2020 MSC: 15A04, 93C95, 93C99

KEYWORDS: Vector control, field oriented control, Direct torque control, Permanent magnet synchronous motor, Asynchronous motor

Introduction

Electric motors are fundamental components in various industrial applications, from automotive propulsion to renewable energy systems. Achieving optimal motor performance, efficiency, and reliability is essential for enhancing overall system efficiency and reducing energy consumption. Vector control algorithms play a crucial role in achieving these objectives by precisely controlling the motor's stator currents and magnetic fields.

In recent years, significant advancements have been made in motor control technologies, enabling more sophisticated and efficient vector control strategies. The application of vector control algorithms has gained traction due to their ability to improve motor drive system performance under varying load conditions and dynamic operating scenarios [1].

Vector transformation

Modeling and mathematical transformations of an electric motor are used to solve equations by orienting all variables to a common reference frame. Clarke Transform is used to convert balanced three-phase quantities into balanced two-phase quantities. Consequently, the control of the two-phase system can be provided in a simpler way, such as DC motors [6]. Park Transformation is used to convert balanced three phase vectors to an orthogonal rotating reference frame. Formulation of Park Transform, Clarke Transform, and Inverse Park Transform are given with Equation 1, 2 and 3 respectively.

$$\begin{bmatrix} d \\ q \\ 0 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} \sin(\theta) & \sin(\theta - {}^{2\pi}/_{3}) & sin(\theta + {}^{2\pi}/_{3}) \\ \cos(\theta) & \cos(\theta - {}^{2\pi}/_{3}) & \cos(\theta + {}^{2\pi}/_{3}) \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} f_\alpha \\ f_\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ {}^{1}/_{\sqrt{3}} & {}^{2}/_{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} f_a \\ f_b \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} f_\alpha \\ f_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} f_d \\ f_q \end{bmatrix}. \tag{3}$$

Permanent magnet synchronous motors (PMSM)

Permanent magnet synchronous motors (PMSM) are preferred in many industrial applications for reasons such as providing high torque at low speeds, high power density and high efficiency. The mathematical model of PMSM is usually defined by the following equations [3]:

$$\psi_{sd} = L_{sd} i_{sd} + \Psi \tag{4}$$

$$\psi_{sq} = L_{sq} i_{sq} \tag{5}$$

$$v_{sd} = R_s i_{sd} + L_{sd} \frac{di_{sd}}{dt} - \omega_r L_{sq} i_{sq} \tag{6}$$

$$v_{sq} = R_s i_{sq} + L_{sq} \frac{di_{sq}}{dt} + \omega_r \left(L_{sd} i_{sd} + \Psi \right) \tag{7}$$

$$T_e = \frac{3}{2} P \left(\psi_{sd} i_{sq} - \psi_{sq} i_{sd} \right), \tag{8}$$

where ψ_{sd}, ψ_{sq}, v_{sd}, v_{sq}, i_{sd} and i_{sq}, symbols represent motor fluxes, voltages and currents, ω_r is electrical angular velocity, T_e is electromagnetic torque in $d{-}q$ coordinates, respectively. Regarding the motor parameters, Ψ is the flux of the permanent magnet, P is the number of pole pairs, R_s is the stator resistance, and the stator inductance can be divided into two different components, L_{sd} and L_{sq}, due to the characteristics of PMSM. The model is completed with the mechanical equation defined as:

$$J \frac{d\omega_m}{dt} = T_e - T_l - B\omega_m \tag{9}$$

$$\omega_r = P\omega_m \tag{10}$$

$$T_e = \frac{3}{2} P \Psi i_{sq}, \tag{11}$$

where J represents the inertia of the motor and the connected load, T_l is the load torque, B is the coefficient of friction and ω_m is the mechanical angular velocity.

Asynchronous motor

Asynchronous motors are preferred due to their simple and robust structure, low cost and minimum maintenance requirement. Asynchronous motors dominate the market today and have replaced DC motors in high performance applications where variable speed and torque control is required. Mathematical modeling of the asynchronous motor can be implemented with the following formulas to calculate torque and flux of the motor [8].

$$V_s = R_s I_s + \frac{d}{dt} \psi_s \tag{12}$$

$$0 = R_r I_r + \frac{d}{dt} \psi_s - j\omega_r \psi_r \tag{13}$$

$$\psi_s = L_s I_s + L_m I_r \tag{14}$$

$$\psi_r = L_r I_r + L_m I_s \tag{15}$$

$$\psi_{sd} = \int (V_{sd} - R_s I_{sd})\, dt \tag{16}$$

$$\psi_{sq} = \int (V_{sq} - R_s I_{sq})\, dt \tag{17}$$

$$|\psi_s| = \sqrt{{\psi_{sd}}^2 + {\psi_{sq}}^2} \tag{18}$$

$$\theta_{\psi_s} = \tan^{-1}\left({\psi_{sq}}/{\psi_{sd}}\right) \tag{19}$$

$$T_e = \frac{3}{2}\frac{P}{2}\left(\psi_{sq} I_{sd} - \psi_{sd} I_{sq}\right), \tag{20}$$

where θ_{ψ_s} represents angle of the flux and other parameter defined in PMSM section.

Field oritented control (FOC)

The first study with FOC belongs to Blachke and studied in 1971. In that study, the effect of FOC on asynchronous motors was investigated [2].

In FOC, three-phase stator currents are reduced to two phase reference frame with Clarke transform. Then currents are converted to the dq axis with the Park transform using rotor angle of position sensor. d axis stator current controls the flux and, similarly, q axis current controls the torque.

The d loop control with PI (proportional - integral) regulator aims to minimize d axis current error. q loops are connected in cascade structure. The outer loop contains speed control PI regulator that controls the motor speed and PI output gives the torque reference value. The inner loop controls i_{sq} current to reach desired torque reference.

Direct torque control (DTC)

The first study with DTC belongs to Takahashi and studied in 1986 [7]. DTC estimates stator flux and motor torque by using the modelling equations of the motor. Then, estimated values are used for seperate control of the flux and the torque. Without a position sensor, this estimation can be performed with two stator phase currents, two stator voltages [11]. In classical DTC this estimation is based on the integration of the stator voltage equation:

$$\psi_s = \int (V_s - R_s i_s\ )\, dt. \tag{21}$$

The change in stator flux is taken to be directly proportional to the applied stator voltage, ignoring the voltage drop across the stator resistance. Thus, torque can be controlled by rapidly changing the stator flux position through the stator voltage applied to the motor.

In DTC, the motor current is measured and then converted to the dq axis with the conversion formulas. Depending on the type of motor (asynchronous, synchronous, etc.), the necessary mathematical models are defined and the stator flux and torque are estimated. Flux and torque are compared with reference values and error values are calculated. Voltage vectors are determined according to the sign of the fault, each voltage vector represents the state of the switching elements. The lookup table given in [7], which is a general table used for DTC checking.

Conclusion and discussion

PMSM is used in applications where precise control is required. In comparison with asynchronous motors, permanent magnet motors are more efficient due to their permanent magnet structure. For these reasons, it is more expensive than asynchronous motors. Asynchronous motors adapt better to load changes and are currently the most used motor type in the market, they are both economical and widely available.

FOC is generally preferred control strategy for synchronous motors. Both precise position control and speed control can be succeed with FOC in PMSM motors. It is used in applications that require precise control, for example CNC machines, robots and medical equipment [9, 4, 5]. On the other hand, FOC has some problem like system instability due to possible changes in motor parameters or disturbances during operation, as well as the large number of calculations that limit the control speed [1].

In applications where speed control is not required but torque control is required, direct torque control is much easier to implement. They have a simpler structure than the FOC, they do not require the use of PWM (Pulse Width Modulation), speed or position sensors. That's why DTC is generally preferred control strategy for asynchronous motors. DTC does not consist of cascade loops as in FOC. Such systems have dynamic characteristics, respond quickly to load changes and are less sensitive to changes and distortions in engine parameters. However, DTC is known to have more current and torque ripple compared to FOC.

The response of PMSM motor to different vector control algorithms [1] and the response of asynchronous motor to different vector control algorithms [10] are frequently studied. In this study, both mathematical modeling of motor types and control algorithms are given. According to models, the features of the motors were compared and the control methods compatible with the motor types were specified. In our future study, these comparisons will be examined with the simulation results.

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Dickson collocation method to solve first-order differential equations which variable delays

Suayip Yuzbasi [*1] *and Ozlem Karaagacli* [2]

In this paper, a collocation method based on Dickson polynomials is presented to solve systems of first-order linear equations with variable delay. The assumed solution in form of the truncated Dickson series and its derivatives are expressed in matrix forms. By using the necessary matrix forms and collocation points, the problem is reduced to a system of algebraic linear equations. Numerical application is made to show the effectiveness and applicability of the technique. The results are expressed in tables.

2020 MSC: 34A30, 65L60, 34K09

KEYWORDS: Dickson collocation method, Dickson polynomials, Delay differential equations, Variable delays

Introduction

Differential equations are widely used in many fields of science. Numerical methods are used when it is not possible to calculate exact solutions of these equations. Many studies have been done on linear and delay differential equations over the years. First order linear differential equations was examined by collocation method based on Laguerre polynomials [7]. The solutions of linear differential equations was approximated with Morgan-Voyce polynomials [2]. Many methods on approximate solutions of delayed differential equations have been presented [3, 8, 9, 10]. Many equations was studied using Dickson polynomials [1, 4, 5, 6].

In this study, we present non-homogeneous differential equation with variable delays and variable coefficients in the form

$$u'(t) = f(t) + Z_1(t)u(t) + \sum_{i=2}^{c} Z_i(t)u(t - \mu_i(t)) \tag{1}$$

with initial condition

$$u(a) = \gamma. \tag{2}$$

Here, $u(t)$ is unknown function, $f(t)$ and $Q_i(t)$, $i = 2, 3, \ldots, c$ are coefficients, $\mu_i(t)$ are non-negative delays in the interval $0 \leq a \leq t \leq b$.

Our aim in this study is to find the approximate solution to the problem (1) under initial condition (2) in the form of the following truncated Dickson series

$$u_N(t) = \sum_{n=0}^{N} a_n D_n(t). \tag{3}$$

Here, $N \geq 1$ is selected to be any positive integer. $a_{i,n}$, $n = 0, 1, 2, \ldots, N$ are unknown Dickson coefficients and $D_n(x, \alpha)$, $n = 0, 1, 2, \ldots, N$ are the Dickson polynomials defined by [4]

$$D_n(x,\alpha) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{n}{n-p} \binom{n-p}{p} (-\alpha)^p x^{(n-2p)}, \quad n \geq 1, \quad -\infty < x < \infty.$$

Fundamental matrix relations

We can write the matrix form of Dickson polynomials as [4]

$$\mathbf{D}(t,\alpha) = \mathbf{T}(t)\mathbf{S}(\alpha) \tag{4}$$

where [4]

$$\mathbf{D}(t,\alpha) = \begin{bmatrix} D_0(t,\alpha) & D_1(t,\alpha) & \ldots & D_N(t,\alpha) \end{bmatrix}$$

and

$$\mathbf{T}(t) = \begin{bmatrix} 1 & t & t^2 & \ldots & t^N \end{bmatrix}.$$

If N is even, [4]

$$\mathbf{S}^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{N}{\frac{N}{2}}\binom{\frac{N}{2}}{\frac{N}{2}}(-\alpha)^{\frac{N}{2}} & 0 & \frac{N}{\frac{N}{2}+1}\binom{\frac{N}{2}+1}{\frac{N}{2}-1}(-\alpha)^{\frac{N}{2}-1} & 0 & \cdots & \frac{N}{N}\binom{N}{0}(-\alpha)^0 \end{bmatrix}.$$

If N is odd, [4]

$$\mathbf{S}^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{N}{\lceil\frac{N}{2}\rceil}\binom{\lceil\frac{N}{2}\rceil}{\lfloor\frac{N}{2}\rfloor}(-\alpha)^{\lfloor\frac{N}{2}\rfloor} & 0 & \frac{N}{\lceil\frac{N}{2}\rceil+1}\binom{\lceil\frac{N}{2}\rceil+1}{\lfloor\frac{N}{2}\rfloor-1}(-\alpha)^{\lfloor\frac{N}{2}\rfloor-1} & \cdots & \frac{N}{N}\binom{N}{0}(-\alpha)^0 \end{bmatrix}.$$

The matrix form of equation (3) can be expressed as

$$u_N(t) = \mathbf{D}(t,\alpha)\mathbf{A}. \tag{5}$$

If we substitute Eq. (4) in Eq. (5), we get

$$u_N(t) = \mathbf{T}(t)\mathbf{S}(\alpha)\mathbf{A}. \tag{6}$$

If we write the matrix form of the derivative of $u_N(t)$, we have

$$u_N^{(1)}(t) = \mathbf{D}^{(1)}(t,\alpha)\mathbf{A} \tag{7}$$

or from Eq. (6)

$$u_N^{(1)}(t) = \mathbf{T}^{(1)}(t)\mathbf{S}(\alpha)\mathbf{A},$$

where

$$\mathbf{T}^{(1)}(t) = \mathbf{T}(t)\mathbf{B}.$$

Hence, we have

$$u_N^{(1)}(t) = \mathbf{T}(t)\mathbf{B}\mathbf{S}(\alpha)\mathbf{A} \tag{8}$$

where [4]

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix}^T.$$

If $t - \mu_i(t)$ is written instead of t in equation (6), we get the following relation

$$u_N(t - \mu_i(t)) = \mathbf{T}(t - \mu_i(t))\mathbf{S}\mathbf{A}. \tag{9}$$

Here, $\mathbf{T}(t - \mu_i(t))$ is equal to

$$T(t - \mu_i(t)) = \mathbf{T}(t)\mathbf{K}(-\mu_i(t)) \tag{10}$$

where

$$\mathbf{K}(-\mu_i(t)) = \begin{bmatrix} \binom{0}{0}(-\mu_i(t))^0 & \binom{1}{0}(-\mu_i(t))^1 & \binom{2}{0}(-\mu_i(t))^2 & \dots & \binom{N}{0}(-\mu_i(t))^N \\ 0 & \binom{1}{1}(-\mu_i(t))^0 & \binom{2}{1}(-\mu_i(t))^1 & \dots & \binom{N}{1}(-\mu_i(t))^{N-1} \\ 0 & 0 & \binom{2}{2}(-\mu_i(t))^0 & \dots & \binom{N}{2}(-\mu_i(t))^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{N}{N}(-\mu_i(t))^0 \end{bmatrix}.$$

Substituting equation (10) in equation (9), we achieve

$$u_N(t - \mu_i(t)) = \mathbf{T}(t)\mathbf{K}(-\mu_i(t))\mathbf{S}\mathbf{A}. \tag{11}$$

If we replace the relations (6), (8) and (11) into equation (1), we can express the matrix form as

$$\Big\{\mathbf{T}(t)\mathbf{B}\mathbf{S}(\alpha) - Z_1(t)\mathbf{T}(t)\mathbf{S}(\alpha) - \sum_{i=2}^{c} Z_i(t)\mathbf{T}(t)\mathbf{K}(-\mu_i(t))\mathbf{S}(\alpha)\Big\}\mathbf{A} = f(t). \tag{12}$$

Method of solution

Firstly, we note that we will use the following collocation points

$$t_s = a + \frac{b-a}{N}s, \qquad s = 0, 1, 2, \dots, N. \tag{13}$$

By putting the collocation points obtained from (13) in system (12), we get

$$\Big\{\mathbf{T}(t_s)\mathbf{B}\mathbf{S}(\alpha) - Z_1(t_s)\mathbf{T}(t_s)\mathbf{S}(\alpha) - \sum_{i=2}^{c} Z_i(t_s)\mathbf{T}(t_s)\mathbf{K}(-\mu_i(t_s))\mathbf{S}(\alpha)\Big\}\mathbf{A} = f(t_s). \tag{14}$$

Converting the (14) system to matrix form, it can be written as

$$\Big\{\mathbf{T}\mathbf{B}\mathbf{S}(\alpha) - \mathbf{Z}1\mathbf{T}\mathbf{S}(\alpha) - \sum_{i=2}^{c} \mathbf{Z}_i\tilde{\mathbf{T}}\tilde{\mathbf{K}}(-\mu_i)\mathbf{S}(\alpha)\Big\}\mathbf{A} = \mathbf{F} \tag{15}$$

where

$$\mathbf{T}=\begin{bmatrix}\mathbf{T}(t_0)\\ \mathbf{T}(t_1)\\ \vdots\\ \mathbf{T}(t_N)\end{bmatrix}=\begin{bmatrix}1 & t_0 & t_0^2 & \dots & t_0^N\\ 1 & t_0 & t_1^2 & \dots & t_1^N\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & t_N & t_N^2 & \dots & t_N^N\end{bmatrix},\quad \tilde{\mathbf{T}}=\begin{bmatrix}\mathbf{T}(t_0) & 0 & \dots & 0\\ 0 & \mathbf{T}(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{T}(t_N)\end{bmatrix},$$

$$\mathbf{F}=\begin{bmatrix}f(t_0)\\ f(t_1)\\ \vdots\\ f(t_N)\end{bmatrix},\quad \tilde{\mathbf{K}}(-\mu_i)=\begin{bmatrix}\mathbf{K}(-\mu_i(t_0))\\ \mathbf{K}(-\mu_i(t_1))\\ \vdots\\ \mathbf{K}(-\mu_i(t_N))\end{bmatrix},$$

$$\mathbf{Z}_i=\begin{bmatrix}\mathbf{Z}_i(t_0) & 0 & \dots & 0\\ 0 & \mathbf{Z}_i(t_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{Z}_i(t_N)\end{bmatrix},\quad i=1,2,\dots,p.$$

We can briefly write the fundamental matrix equation (15) of equation (1) as

$$\mathbf{WA}=\mathbf{F}\quad ;\quad [\mathbf{W};\mathbf{F}]$$

where

$$\mathbf{W}=\mathbf{TBS}(\alpha)-\mathbf{Z}1\mathbf{TS}(\alpha)-\sum_{i=2}^{c}\mathbf{Z}_i\tilde{\mathbf{T}}\tilde{\mathbf{K}}(-\mu_i)\mathbf{S}(\alpha).$$

On the other side, the basic matrix form for the condition can be written as

$$\mathbf{MA}=\gamma,\qquad [\mathbf{M};\gamma],$$

where

$$\mathbf{M}=\mathbf{T}(a)\mathbf{S}(\alpha)=[m_{00}\quad m_{01}\quad \dots\quad m_{0N}].$$

By replacing any row of the expanded matrix $[\mathbf{W};\mathbf{F}]$ with the matrix of the condition $[\mathbf{M};\gamma]$, we get

$$\overline{\mathbf{W}}\mathbf{A}=\overline{\mathbf{F}}\quad ;\quad [\overline{\mathbf{W}};\overline{\mathbf{F}}].$$

If $rank\overline{\mathbf{W}}=rank[\overline{\mathbf{W}};\overline{\mathbf{F}}]=(N+1)$, then we can written

$$\mathbf{A}=\overline{\mathbf{W}}^{-1}\overline{\mathbf{F}}.$$

By substituting the coefficients $a_0,a_1,\dots,a_N$ in Eq. (3), we get the Dickson polynomial solution

$$u_N(t)=\sum_{n=0}^{N}a_nD_n(t).$$

Numerical application

Example 1. *We examine first order differential equation with variable delays t^2*

$$2u'(t)-tu(t)+te^{2t^2}u(t-t^2)=4e^{2t},\quad 0\le t\le 1 \tag{16}$$

with initial condition

$$u(0)=1. \tag{17}$$

The exact solution for this problem is e^{2t}. In problem (16), $a = 0$, $b = 1$, $c = 2$, $f(t) = 2e^{2t}$, $Z_1(t) = t/2$, $Z_2(t) = (-te^{2t^2})/2$.

From Eq. (3), by choosing $N = 3$, we obtain the truncated Dickson series form for this problem

$$u_N(t) = \sum_{n=0}^{3} a_n D_n(t).$$

If we calculate the collocation points by choosing N value 3 according to Eq. (13), we get

$$\left\{ t_0 = 0, \quad t_1 = \frac{1}{3}, \quad t_2 = \frac{2}{3}, \quad t_3 = 1 \right\}.$$

From Eq. (15), the fundamental matrix equation for this problem is written as

$$\left\{ \mathbf{TBS}(\alpha) - \mathbf{Z}_1\mathbf{TS}(\alpha) - \mathbf{Z}_2\tilde{\mathbf{T}}\tilde{\mathbf{K}}(-\mu_2)\mathbf{S}(\alpha) \right\}\mathbf{A} = \mathbf{F}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}(t_0) \\ \mathbf{T}(t_1) \\ \mathbf{T}(t_2) \\ \mathbf{T}(t_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{T}(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix},$$

$$\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{T}(0) & 0 & 0 & 0 \\ 0 & \mathbf{T}(1/3) & 0 & 0 \\ 0 & 0 & \mathbf{T}(2/3) & 0 \\ 0 & 0 & 0 & \mathbf{T}(1) \end{bmatrix},$$

$$\mathbf{Z_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \mathbf{Z_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-e^{2/9}}{6} & 0 & 0 \\ 0 & 0 & \frac{-e^{8/9}}{3} & 0 \\ 0 & 0 & 0 & \frac{-e^2}{2} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 2 \\ 2e^{2/3} \\ 2e^{4/3} \\ 2e^2 \end{bmatrix}, \quad \mathbf{S}^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{K}}(-\mu_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1/9 & 1/81 & -1/729 \\ 0 & 1 & -2/9 & 1/27 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 1 \\ 1 & -4/9 & 16/81 & -64/729 \\ 0 & 1 & -8/9 & 16/27 \\ 0 & 0 & 1 & -4/3 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By performing operations, $[\mathbf{W};\mathbf{F}]$ is found as

$$[\mathbf{W};\mathbf{F}]=\begin{bmatrix} 0 & 1 & 0 & 0 & ; & 2 \\ \frac{e^{2/9}}{3}-\frac{1}{3} & \frac{e^{2/9}}{27}+\frac{17}{18} & \frac{2e^{2/9}}{243}+\frac{35}{54} & \frac{4e^{2/9}}{2187}+\frac{53}{162} & ; & 2e^{2/3} \\ \frac{2e^{8/9}}{3}+\frac{2}{3} & \frac{2e^{8/9}}{27}+\frac{7}{9} & \frac{4e^{8/9}}{243}+\frac{32}{27} & \frac{8e^{8/9}}{2187}+\frac{100}{81} & ; & 2e^{4/3} \\ e^2-1 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & ; & 2e^2 \end{bmatrix}. \tag{18}$$

The expanded matrix for the initial condition is found as

$$[\mathbf{M};\gamma]=\begin{bmatrix} 2 & 0 & 0 & 0 & ; & 1 \end{bmatrix}. \tag{19}$$

By replacing the expanded matrix (19) with the last row of the matrix (18) we get the new expanded matrix as

$$[\mathbf{W};\mathbf{F}]=\begin{bmatrix} 0 & 1 & 0 & 0 & ; & 2 \\ \frac{e^{2/9}}{3}-\frac{1}{3} & \frac{e^{2/9}}{27}+\frac{17}{18} & \frac{2e^{2/9}}{243}+\frac{35}{54} & \frac{4e^{2/9}}{2187}+\frac{53}{162} & ; & 2e^{2/3} \\ \frac{2e^{8/9}}{3}+\frac{2}{3} & \frac{2e^{8/9}}{27}+\frac{7}{9} & \frac{4e^{8/9}}{243}+\frac{32}{27} & \frac{8e^{8/9}}{2187}+\frac{100}{81} & ; & 2e^{4/3} \\ 2 & 0 & 0 & 0 & ; & 1 \end{bmatrix}.$$

Solving the system, the Dickson coefficients matrix $\mathbf{A}$ is obtained as

$$\mathbf{A}=\begin{bmatrix} 1/2 \\ 2 \\ 1.487380498043678 \\ 2.711429774823984 \end{bmatrix}.$$

By substituting the matrix $\mathbf{A}$ in Eq. (6) , the approximate solution of $u_3(t)$ is gained as

$$u_3(t)=2.711429774823984t^3+1.487380498043677t^2+2t+1.$$

Similarly, other solutions are obtained by applying the method to $N=4$ and $N=5$.

$$u_4(t)=1.458188294981660t^4+0.784863146280799t^3+2.118283310850131t^2$$
$$+2t+1.000000000000057,$$
$$u_5(t)=0.610519921806230t^5+0.323343621365069t^4+1.472276766171916t^3$$
$$+1.979363301797854t^2+2t+1.$$

In Table 1, the numerical results of u obtained are compared with the exact solution of the problem and the solutions obtained by the Laguerre collocation method [7]. In Table 2, the comparison of error analyzes in solving the problem with the Laguerre collocation method [7] is given. Table 3 presents the actual absolute errors obtained for different α values in the Dickson collocation method.

	Exact Solution	Present Method		LCM [7]	
t_i	$u(t_i)=e^{2t}$	$u_3(t_i)$	$u_5(t_i)$	$u_3(t_i)$	$u_5(t_i)$
0	1	1	1	1	1
0.2	1.491824697641270	1.481186658120339	1.491665462370451	1.481186658120339	1.491665462370452
0.4	2.225540928492468	2.211512385275723	2.225453162028894	2.211512385275724	2.225453162028898
0.6	3.320116922736547	3.321125810657704	3.319961932588919	3.321125810657705	3.319961932588913
0.8	4.953032424395115	4.940175563457835	4.953094932719223	4.940175563457834	4.953094932719199
1	7.389056098930650	7.198810272867664	7.385503611141019	7.198810272867663	7.385503611140948

Table 1: Comparision of the exact solutions and approximate solutions of Eq. (16) for $N=3,5$

	Present Method			LCM [7]		
t_i	$e_3(t_i)$	$e_4(t_i)$	$e_5(t_i)$	$e_3(t_i)$	$e_4(t_i)$	$e_5(t_i)$
0	0	0	0	0	0	0
0.2	1.0638e-02	1.5186e-03	1.5924e-04	1.0638e-02	1.5186e-03	1.5924e-04
0.4	1.4029e-02	9.4526e-04	8.7766e-05	1.4029e-02	9.4526e-04	8.7766e-05
0.6	1.0089e-03	9.7671e-04	1.5499e-04	1.0089e-03	9.7671e-04	1.5499e-04
0.8	1.2857e-02	1.7928e-03	6.2508e-05	1.2857e-02	1.7928e-03	6.2508e-05
1	1.9025e-01	2.7721e-02	3.5525e-03	1.9025e-01	2.7721e-02	3.5525e-03

Table 2: Comparision of actual absolute error of Eq. (16) for $N = 3, 4$ and 5

	Present Method					
t_i	$e_3(t_i)$ for $\alpha = -2$	$e_3(t_i)$ for $\alpha = 3$	$e_4(t_i)$ for $\alpha = -2$	$e_4(t_i)$ for $\alpha = 3$	$e_5(t_i)$ for $\alpha = -2$	$e_5(t_i)$ for $\alpha = 3$
0	0	0	0	0	-1.8190e-12	-6.3665e-12
0.2	1.0638e-02	1.0638e-02	1.5186e-03	1.5186e-03	1.5924e-04	1.5924e-04
0.4	1.4029e-02	1.4029e-02	9.4526e-04	9.4526e-04	8.7766e-05	8.7766e-05
0.6	1.0089e-03	1.0089e-03	9.7671e-04	9.7671e-04	1.5499e-04	1.5499e-04
0.8	1.2857e-02	1.2857e-02	1.7928e-03	1.7928e-03	6.2508e-05	6.2508e-05
1	1.9025e-01	1.9025e-01	2.7721e-02	2.7721e-02	3.5525e-03	3.5525e-03

Table 3: Comparision of actual absolute error of Eq. (16) for $N = 3, 4, 5$, $\alpha = 0, -2$ and 3

Conclusion

In this article, a collocation method based on Dickson polynomials for first-order differential equations with variable delay is presented. An example is given in the fourth section to demonstrate the validity of the method. Table 1 shows that the obtained results are close to the exact solution. At the same time, it is seen that similar results are obtained with the solutions obtained by the Laguerre collocation method [7] in Table 1. Looking at Table 2, the numerical results of the actual absolute error obtained by the current method and the actual absolute error obtained by the Laguerre collocation method [7] are similar for $N = 3$, $N = 4$ and $N = 5$. As the N value increases, the actual absolute error decreases. Table 3 shows the actual absolute error comparisons obtained by the presented method for different values of N and α. It can be seen that similar results are obtained even if the α value changes.

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Using multivariate statistical analyzes in geochemical data: Example of lead-zinc deposit

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Geostatistical methods are frequently used in the geochemical evaluation of mineral deposits. Geostatistical methods are preferred by researchers for reasons such as less error rates than traditional methods and being based on precise results. The most commonly used geostatistical methods are as follows: means, measures of variability, classification, regression and correlation analysis, analysis of variance, correlation function (variogram), estimation error variance, kriging and simulation. The aim of this study is to emphasize the importance of geostatistical methods in the evaluation of geochemical analyzes during the exploration and operation of mineral deposits. It is within the scope of the study to highlight the details of the multivariate data analysis, which is mostly preferred for geochemical analyzes containing large numbers of data, with examples in the literature. The error rates of traditional geometric methods such as polygon, triangle and section used in the calculation of reserve and grade of mineral deposits are higher than the error rates of the values calculated using geostatistical methods. In this respect, it is suggested as a result of the study that researchers who will work on the evaluation of mineral deposits should first turn to geostatistical methods.

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Introduction

Geostatistical methods are a type of applied statistics used for the interpretation of geochemical analyzes and estimation of problems [17]. The foundations of the theory were laid by the French mining engineer G. Matheron [17]. Geostatistical interpretation methods are being applied by researchers by expanding them. When geostatistical methods are used for the evaluation of mineral deposits, it is an important method used in many aspects such as determining the extent of mineralization, monitoring and quality extraction. It is possible to gather the main geostatistics under two main roofs: Random variables and data analysis. Data analysis from a geoscience perspective is the most common geostatistical method for the evaluation of mineral deposits. Multivariate data analysis constitutes the first step of the project by providing accurate summary and interpretation of geochemical data [17]. The purpose of factor analysis, which is one of the multivariate statistical analyzes, is to reduce the high number of variables to a small number and to make the data easy to understand [1], [7], [18]. As a result of factor analysis, a common variance value is learned [1], [14]. In this study, the benefits of data analysis, which is one of the geostatistical methods, in the evaluation of mineral deposits are emphasized. Data analysis covers the determination of the basic statistical properties of the results of geochemical analysis containing a large number of data and presenting them in a summary form. Histograms, one of the graphical statistical methods, are the sample counterpart of the density function of the random variable. However, while they allow graphical representations, they do not display descriptive statistical data numerically. For this purpose, descriptive statistical analyzes are also performed. Descriptive statistics are divided into three groups: Measures of central tendency (Arithmetic mean, Geometric mean, Harmonic mean, Mode, Median); Measures of variability (range of variation, mean absolute deviation, variance, standard deviation, coefficient of variation); Distribution shape measures (skewness measures, Pearson asymmetry measures, Bowley asymmetry measures, Kurtosis and Skewness measures) [6]. In other words, descriptive statistics are divided into three as location, spread and shape statistics. Location statistics describe the center of the distribution. Diffusion statistics express the degree of variability of the data. Shape-measuring statistics measure the tail length of distribution types. The common feature of all descriptive statistics is that they simply present the data numerically. Histogram plots and relational lineage plots divide the data into specific class ranges, and it is possible to subtract the number of data groups in each class range and obtain the frequency value [10], [11], [12], [13]. Geometric traditional methods such as polygon, triangle and section are used in the calculation of mineral deposits reserve. The basis of these methods is based on geostatistics. Considering a mineral deposit, the difference between the grade values will be f(x)=z(a)-z(a+x) when the grade value of the ore at point a is indicated by the grade value z(a+x) at a distance x from here. Calculation and modeling of the function of distance X will be important in reserve calculation methods. However, this relationship was later destroyed by the theory of regional variables [8], [5]. In geostatistics, the grade changes of regional variables with distance are modeled as stationary random functions. A few of the studies dealing with geostatistical methods in the evaluation of mineral deposits are summarized as follows; Geological modeling of the polymetallic mineral deposit containing Pb, Zn and Ag in Balikesir-Balya Hastanesitepe locality was made by simulation and Kriging methods. Simulation method was used for the distribution of host rock and ore at certain elevations of the underground. However, they thought that the simulation was weak in showing the subterranean melting spaces (lens, karstic void, alteration zone, etc.) [3]. The data obtained as a result of the geochemical analyzes on the urban soils of the Brazilian Parana region were evaluated geostatistically. The Matheron Semivariogram method was used on the normally distributed data. The semivariogram was made according to 3 theoretical models: spherical, exponential and Gaussian. The study showed that geostatistical analysis can make good predictions for trace metals [2]. The physical and chemical properties of the soils of the Bindiba mining area in eastern Cameroon were analyzed. Forecast maps at unsampled locations in the region were extracted by variographic analysis and normal Krigging (OK) geostatistics methods. The spatial variability of the metal concentrations of the region was learned with the maps drawn from the geostatistical models [9]. Triangular prism, trapezoid, polygon, isopaque, etc. for reserve calculations of ore deposits planned to be operated and produced. Traditional methods are often preferred. However, it has been observed that the error rates of the predictions made by traditional methods are at high levels. Contrary to these methods, the reasons why geostatistical modeling techniques are more reliable than traditional methods can be summarized as follows: it provides estimation and simulation in unsampled areas, the continuity, continuity and anisotropy of the ore bed are measured in the light of the estimations, suitable location estimates are made for sampling, the grade distribution of the ore in the deposit is made, Error rates of predictions and simulations are calculated, and their reliability and validity are given with precise results. In the literature, there are studies on the interpretation of geochemical analyzes made during the exploration and operation of mineral deposits using different geostatistical methods. However, no research has been found that emphasizes the importance of geostatistical methods in the evaluation of mineral deposits and addresses all methods. In this context, the aim of this research is to touch on all geostatistical methods that can be used in the evaluation of mineral deposits and to emphasize their importance.

Main results

The results of the geochemical analysis of the lead-zinc deposit in the Silesian-Cracow region in Southern Poland were interpreted by statistical analysis. Some descriptive statistical data (minimum, maximum, standard deviation, variance, coefficient of variation) of ore samples and wall rock (dolomite) samples collected from the study area were compared with the anomaly values known in the literature such as earth crust, continental crust, carbonate rocks. The cumulative frequency distribution curve was drawn on the logarithmic probability paper of the zinc concentrations of the samples collected from different locations. Here, it was thought that the zinc was close to the lognormal. Cumulative frequency distribution curves were also drawn for zinc and lead. It has been discussed whether the samples collected from different locations of the region are within the operable limits. The histogram of zinc in the content of the dolomite samples, which is the country rock of the region and found with the ore, was drawn. In the study, statistical analyzes of lead-zinc content in dolomite made it easier to distinguish the threshold values of ore bodies [15]. The geochemical analysis data of the Mehdiabad zinc-lead deposit in Iran has been interpreted geostatistically. Correlation and regression analysis were applied to the data that were adjusted to normal distribution. Elements were categorized by principal component matrix (PCA) analysis method. The distribution of the selected elements is visualized by the 3D modeling method. In this study, Pearson correlation analysis and eigenvalues/eigenvectors covariance analysis were applied. As a result of the study, the geostatistics results of the samples collected from different locations divided the elements into three main categories. 1. For the study, Si, Al, Na, Ni, P, Ti, Ce, Cr, La, Li, V and Zr elements are abundantly found in the deep parts of the ore deposit in the structure of quartz, ilmenite, zircon, monazite and clay group minerals. 2. Elements forming the side rock of the ore, Ca and Mg, were chemically deposited in the limestone and parent rocks. 3. They are hydrothermal fluids that bring the ore-forming elements Pb, S, Zn, Ag and Cd. With the results of the factor analysis, it was seen that Fe, Ba and Mn had a different origin. It is thought to have a different origin from the hydrothermal fluids that make up the ore [1].

The geochemical data of lead-zinc in the metallogenic belt in Kerman region of Iran are statistically explained. Analysis data has been normalized. Variograms of Pb and Zn concentrations were extracted for each exploration well. In addition, the behavior of Pb and Zn was tried to be measured precisely with the K-means clustering method. As a result of the K-means clustering analysis, it was determined that Pb and Zn exhibited geochemically similar linear behavior [4]. The Zn geochemical analysis values of 50 ore samples collected from the lead-zinc deposit of the Bolkardagi (Ulukisla-Nigde) region in Turkiye are given in Table 1 [16]. The descriptive statistical values of the samples are given in Table 2. The histogram graph of the Zn distribution is given in Figure 1-a and the box-plot graph is given in Figure 1-b.

	%Zn		%Zn		%Zn		%Zn		%Zn
1	4.87	**11**	3.25	**21**	5.23	**31**	13.04	**41**	2.87
2	5	**12**	5.74	**22**	5.18	**32**	12.03	**42**	2.99
3	4.87	**13**	5.27	**23**	5.22	**33**	12.24	**43**	2.96
4	5.10	**14**	5.65	**24**	5.00	**34**	11.82	**44**	2.94
5	5.93	**15**	5.61	**25**	5.23	**35**	11.61	**45**	2.82
6	5.43	**16**	9.02	**26**	5.85	**36**	16.06	**46**	2.79
7	6.21	**17**	9.09	**27**	6.12	**37**	16.73	**47**	2.54
8	4.97	**18**	9.13	**28**	6.28	**38**	16.49	**48**	2.07
9	5.18	**19**	8.64	**29**	5.79	**39**	15.69	**49**	2.68
10	4.96	**20**	8.03	**30**	6.14	**40**	16.03	**50**	2.79

Table 1: Zn concentrations of Turkiye Bolkardagi Zn-Pb deposit [16]

Statistics	Results
Mean	6.94
Median	5.52
Mode	2.79[a]
Std. Deviation	4.12
Variance	16.98
Range	14.66
Skewness	1.16
Kurtosis	0.36

a. Multiple modes exist. The smallest value is shown.

Table 2: Descriptive statistical values of Zn concentrations of Turkiye Bolkardagi Zn-Pb deposit

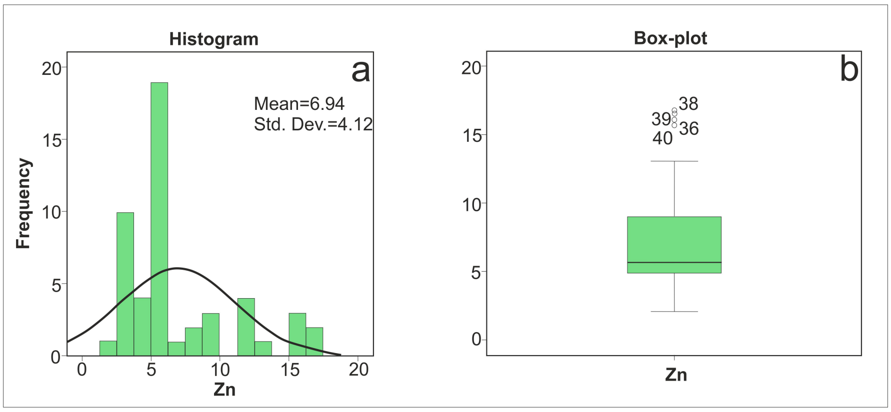

Figure 1: a) histogram plot and b) box-plot plot of Zn distribution

Conclusion

Summary values of 50 Zn samples collected from Bolkardagi zinc lead deposit were presented. The central tendency measures of the descriptive statistics were found as follows, respectively: Arithmetic mean: 6.94; Median (median data when series is ordered from largest to smallest): 5.52; mod (most observed value from series): 2.79.

The results of the scatter shape measurements were found as follows, respectively: Standard deviation (the square root of the mean squared difference of the series from the arithmetic mean): 4.12; variance (square of standard deviation): 4.12; Range (The difference of the maximum value in the series from the minimum value): 14.66. The results of the distribution shape measurements are as follows, respectively: Skewness (can take values in the range of $-\infty$ $+\infty$, if it takes values in the range of ± 3, it is assumed that the distribution is normal): 1.16. Kurtosis (related to the steepness or kurtosis of the scatter plot, value (+) means the curve is steep, (−) means the curve is flat): 0.36. Since the skewness value fell within the range of ± 3, the distribution was considered to be normal. However, the plotted histogram graph showed that the distribution was not normally distributed. According to the histogram graph, the distribution is right skewed and tends to low values. In histogram plots tending to low values, it is thought that there is a relationship between measures of central tendency of the data as follows: Arithmetic mean > Median > Mode. The situation is the same for Bolkardagi Zn samples. Arithmetic mean (6.94) > Median (5.52) > Mode (2.79). The box-plot plot enabled the detection of outliers/outliers in the series. It also made it easier to understand in which range the data were mostly classified. Within the scope of the study, a few simple geostatistical analyzes were tried to be exemplified with the results of the analysis compiled from a previous study from the literature. With geostatistical analysis, it is aimed to summarize the data and make it easier to understand. The importance of geostatistical analysis in the evaluation of mineral deposits has been revealed.

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giα-expansion mappings in topological spaces

Omar Y. Khattab

In this paper, we present a modern expanded generalization of closed sets that are denoted simply by giα-closed sets, and we show the relationship between these new types with other closed sets. Furthermore, we investigate advanced giα-continuous mapping, giα-open mappings giα-irresolute, giα-homeomorphism, and the relationship with properties to each mapping.

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Introduction and preliminaries

We need to expand the open-sets to come true new theorems and corollaries on the homeomorphism mappings and some other advanced mappings, since many researchers that mentioned in the reference introduced huge numbers of closed and open sets. We aim in this work to present other divisions in the family's generalization of closed that are named giα-closed set. We introduce gradually the characteristics of these new types of closed (open) sets and study the relationship with some type of generalized closed (open) sets. We continue to offer some other kinds of mappings for instance, giα-continuous, and giα-open (closed) mapping. Basic continuous mapping led us to discover higher important mappings like giα-irresolute, and giα-homeomorphism. Finally, we succeed to investigate high properties of these mappings. Main while this paper, we suggest to called (X, τ) and (Y, σ) as a topology spaces simply by (resp. $\hat{\tau}$ and $\hat{\sigma}$). We denoted the following definitions for the closure (resp. interior) of a subset B in any topological space by $Cl(B)$ (resp. $Int(B)$), and the complement of B is represented by $\overline{B}$.

Definition 1. *Let $B \subseteq \hat{\tau}$, then B is said to be:*
(i) semi [8] (resp. α [16], regular [9], iα [10])-open set, if $B \subseteq Cl(Int(B))$ (resp. $B \subseteq Cl(Int(Cl(B))), A = Int(cl(A))$, and $B \subseteq Cl(B \cap O)$, where $\exists O \in \alpha O(X), O \neq X, \varnothing$

The collections of all semi (resp. α, regular, and iα-) open sets in $\hat{\tau}$ is symbolized by τ_s (resp. τ_s, τ_r, and $\tau_{i\alpha}$). The complement of open (resp. semi-open, α-open, regular-open, and iα-open) sets in $\hat{\tau}$ called semi (resp. α, regular, and, iα) closed sets. The closure of the above sets is denoted by Cl_s, (resp. Cl_α, Cl_r, and $Cl_{i\alpha}$).

Lemma 2 (*cf.* [10])**.** *Every open (closed) (resp. semi, , and regular) open (closed) set is an iα-open (closed) set.*

Definition 3. *$B \subseteq \hat{\tau}$, then B is said to be:*

(i) g-closed [13] *if $Cl(B) \subseteq O$, such that $B \subseteq O$ and $O \subset \tau$,*

(ii) sg-closed [4] *if $Cl(B) \subseteq O$, such that $B \subseteq O$ and $O \subset \tau_s$,*

(iii) *gs-closed* [2] *if* $Cl_s(B) \subseteq O$, *such that* $B \subseteq O$ *and* $O \subset \tau$,

(iv) *gα-closed* [13] *if* $Cl_\alpha(B) \subseteq O$, *such that* $B \subseteq O$ *and* $O \subset \tau_\alpha$,

(v) *αg-closed* [12] *if* $Cl_\alpha(B) \subseteq O$, *such that* $B \subseteq O$ *and* $O \subset \tau$,

(vi) $r-g$*-closed* [16] *if* $Cl(B) \subseteq O$, *such that* $A \subseteq O$ *and* $O \subset \tau_r$,

(vii) *gαr-closed* [18] *if* $Cl_\alpha(B) \subseteq O$, *such that* $B \subseteq O$ *and* $O \subset \tau_s$.

The collections of g (resp. $sg, gs, g\alpha, \alpha g, r-g$, and $g\alpha r$) open sets are symbolized by τ_g (resp. $\tau_{sg}, \tau_{gs}, \tau_{g\alpha}, \tau_{\alpha g}, \tau_{r-g}$, and $\tau_{g\alpha r}$). The collections of g (resp.$sg, gs, g\alpha, \alpha g, r-g$, and $g\alpha r$) closed that is complement as the followings $\overline{\tau_g}$ (resp. $\overline{\tau_{sg}}, \overline{\tau_{gs}}, \overline{\tau_{g\alpha}}, \overline{\tau_{\alpha g}}, \overline{\tau_{r-g}}$, and $\overline{\tau_{g\alpha r}}$).

Definition 4. *The mapping* $f : \hat{\tau} \to \hat{\sigma}$ *is said to be:*

(i) *g continuous* [5] *(resp. sg-continuous* [6], *gα-continuous* [7], $r-g$*-continuous* [1], *and α-continuous* [10]*) if* $f^{-1}(Q) \subseteq \tau_g$ *(resp.* $\tau_s, \tau_\alpha, \tau_{rg}$, *and* $\tau_{i\alpha}$*),* $\forall Q \subseteq \hat{\sigma}$.

(ii) *iα open* [10], *rg open* [20] *if* $f(Q) \subseteq \sigma_{i\alpha}$, *and* σ_{rg} $\forall Q \subseteq \hat{\tau}$.

(iii) *α-irresolute* [11] *(resp.* $r-g$*-irresolute* [1], *and sg-irresolute* [19]*) if* $f^{-1}(Q) \subseteq \tau_\alpha$ *(resp.* τ_{rg}, *and* τ_s*),* $\forall Q \subseteq \sigma_\alpha$ *(resp.* σ_{rg}, *and* σ_s*).*

Definition 5. *The mapping* $f : \hat{\tau} \to \hat{\sigma}$ *is called g-homeomorphism* [14] *(resp. gc-homeomorphism* [15], *and iα-homeomorphism* [10]*) if f is g (resp. gc and iα−) continuous and open.*

giα-closed set and its applications

Definition 6. *A subset* $B \subseteq \hat{\tau}$ *is said to be generalized iα-closed set simply (giα-closed) se* $Cl_{i\alpha}(B) \subseteq O$, *such that* $B \subseteq O$, $\wedge$ $O \subset \tau_{i\alpha}$.

Example 7. $X = \{0, 1, 2\}$, $\tau = \{\varnothing, \{0\}, X\}$,
$\tau\_i\alpha = \{\varnothing, \{0\}, \{2\}, \{1\}, \{0, 2\}, \{0, 1\}, \{1, 2\}, X\}$. *The complement of giα-closed is giα-open set, we symbolized the collections of giα-open by* $\tau_{gi\alpha}$

Theorem 8. *If* $C \subseteq \hat{\tau}$ *is closed, then B giα-closed.*

Proof. Let $C \subseteq \overline{\hat{\tau}}$, such that $B \subseteq U$, and $U \subseteq \tau_{i\alpha}$. Since $Cl_{i\alpha}(C) \subseteq Cl(C) = C$. It implies $Cl_{i\alpha}(C) \subseteq U$. Therefore C is *giα*-closed set in $\hat{\tau}$. ☐

Corollary 9. *If* $C \subseteq \hat{\tau}$ *is iα-closed, then C is giα-closed set.*

Proof. Let $\subseteq \overline{\tau_{i\alpha}}$, such that $C \subseteq U$, where $U \subseteq \tau_{i\alpha}$. Since $Cl_{i\alpha}(C) \subseteq C$. It led to $Cl_{i\alpha}(B) \subseteq U$. That's mean B is *giα*-closed set in τ. ☐

Generally, the converse of above corollary is not True as this example.

Example 10.

$$X = \{0.1, 0.2, 0.3\},$$

$$\tau = \tau_\alpha = \{\varnothing, \{0.2, 0.3\}, X\},$$

$$\tau_{i\alpha} = \{\varnothing, \{0.2\}, \{0.3\}, \{0.1, 0.2\}, \{0.2, 0.3\}, \{0.1, 0.3\}, X\},$$

and iα- closed sets $= \{X, \{0.1, 0.3\}, \{0.1, 0.2\}, \{0.3\}, \{0.1\}, \{0.2\}, X\}$ *we get* $\{0.1\} \subseteq \{0.1, 0.3\}$, $Cl_{i\alpha}\{0.1\} \subseteq \{0.1\}$, *that's showing* $\{0.1\}$ *is giα-closed non iα-closed set.*

Lemma 11. *If C is open (closed) (resp. semi-open (closed), α-open (closed), and regular open (closed), then C is an $i\alpha$-open* [10].

Theorem 12. *If $C \subseteq \hat{\tau}$, then $C \subseteq \tau_{gi\alpha}$.*

Proof. Suppose $\subseteq \hat{\tau}$. Since $\hat{\tau} \subseteq \tau_{i\alpha}$ by Lemma 11, then $C \subseteq \tau_{i\alpha}$, also since $\overline{\tau_{i\alpha}} \subseteq \overline{\tau_{gi\alpha}}$ Corollary 9. Therefore C is $gi\alpha$-closed set. □

Theorem 13. *Every semi-open is $gi\alpha$-open.*

Proof. Let $\subseteq \tau_s\tau$. Since $\tau_s \subseteq \tau_{i\alpha}$ Lemma 11, hence $\overline{C}$ is is $gi\alpha$-closed by Theorem 8. Therefore, C is $gi\alpha$-open. □

Theorem 14. *If V is g-closed, then V is $i\alpha g$-closed.*

Proof. Suppose $V \subseteq \overline{\tau_g}$, such that $Cl(V) \subseteq O$, when $V \subseteq O \subseteq \tau$. Since $\overline{\hat{\tau}} \subseteq \overline{\tau_{gi\alpha}}$ Theorem 8, then $Cl_{i\alpha(V)} \subseteq Cl(V) \subseteq O$, hence $Cl(V) \subseteq O$, and since $\hat{\tau} \subseteq \tau_{i\alpha}$ Lemma 11, then $V \subseteq O \subseteq \tau_{i\alpha}$. Therefore, V is $i\alpha g$-closed. □

Theorem 15. *If V is sg-closed, then V is $gi\alpha$-closed set.*

Proof. Let $V \subseteq \overline{\tau_{sg}}$, consequently $Cl(V) \subseteq O$, such that $V \subseteq O \subseteq \tau_s$. Since $\overline{\hat{\tau}} \subseteq \overline{\tau_{i\alpha}}$ by Lemma 11, then $Cl(V) \subseteq Cl_{i\alpha}(V) \subseteq O$, and since $\tau_s \subseteq \tau_{i\alpha}$ by Lemma 11, then$V \subseteq O \subseteq \tau_{i\alpha}$. Therefore, V is $i\alpha g$-closed. □

Corollary 16. *If V is gs-closed, then V is $i\alpha g$-closed.*

Proof. Same the proof of above theorem. □

Theorem 17. *If V is $g\alpha$-closed, then V is $i\alpha g$-closed.*

Proof. Let $V \subseteq \overline{\tau_{g\alpha}}$, then $Cl(V)_\alpha \subseteq O$, such that $V \subseteq O \subseteq \tau_\alpha$. Since $\overline{\tau_\alpha} \subseteq \overline{\tau_{i\alpha}}$ by Lemma 2.5, then $Cl(V)_\alpha Cl_{i\alpha} \subseteq O$, also since $\tau_\alpha \subseteq \tau_{i\alpha}$ by Lemma 11, then $V \subseteq O \subseteq \tau_{i\alpha}$. Therefore, V is $i\alpha g$-closed. □

Corollary 18. *Every αg- closed is $i\alpha g$-closed.*

Proof. Same the proof of the above Theorem 17. □

The following example shows that $gi\alpha$-closed set isn't an essential to be g (resp. $gs, sg, \alpha g$, and $g\alpha$) closed sets.

Example 19. *$X = \{5,7,9\}$, $\tau = \{\varnothing, \{7\}, X\}$ $\tau_\alpha = \tau_s = \{\varnothing, \{7\}, \{7,5\}, \{7,9\}, X\}$, $\tau_{i\alpha} = \{\varnothing, \{5\}, \{7\}, \{9\}, \{7,5\}, \{7,9\}, \{9,7\}, X\}$ = $i\alpha$-closed sets. We get $\{9,7\} \subseteq \{9,7\}, Cl_{i\alpha}\{9,7\} \subseteq \{9,7\}$. Therefore $\{9,7\}$ is $gi\alpha$-closed but is not g (resp. $gs, sg, \alpha g$, and $g\alpha$) closed sets.*

Theorem 20. *If V is rg-closed, then V is $gi\alpha$-closed set.*

Proof. Let $V \subseteq \overline{\tau_{rg}}$, so we have $Cl(V) \subseteq O$, such that $V \subseteq O \subseteq \tau_r$. Since $\overline{\hat{\tau}} \subseteq \overline{\tau_{i\alpha}}$ by Lemma 11, then $Cl(V) \subseteq Cl_{i\alpha} \subseteq O$, also $\tau_r \subseteq \tau_{i\alpha}$ by Lemma 11, then $V \subseteq O \subseteq \tau_{i\alpha}$. Therefore, V is $i\alpha g$-closed. □

Corollary 21. *If V is αrg-closed, then V is $gi\alpha$-closed set.*

Proof. Let V is a regular-closed, so we have $Cl_\alpha(V) \subseteq O$, such that $V \subseteq O \subseteq \tau_r$. Since every α closed is $i\alpha$-closed [10], then $Cl_\alpha(V) \subseteq Cl_{\alpha i} \subseteq O$, and since $\tau_r \subseteq \tau_{i\alpha}$ Lemma 11, then $V \subseteq O \subseteq \tau_{i\alpha}$. Therefore, V is $i\alpha g$-closed. □

The following example shows that $gi\alpha$-closed is not a necessary to be αgr-closed, rg-closed, and set αrg-closed as the next example.

Example 22. *$X = \{3,5,7\}$, $\tau = \{\varnothing, \{7\}, X\}$, $\tau_\alpha = \{\varnothing, \{7\}, \{3,7\}, \{5,7\}, X\}$, $\tau_{i\alpha} = \{\varnothing, \{3\}, \{5\}, \{7\}, \{3,5\}, \{3,7\}, \{5,7\}, X\} = i\alpha$- closed sets. We get $\{3,7\} \subseteq \{3,7\}, Cl_{i\alpha}\{3,7\} \subseteq \{3,7\}$. Therefore $\{3,7\}$ is $gi\alpha$-closed but is not αgr-closed, rg-closed, and set αgr-closed.*

Theorem 23. *$\tau_{i\alpha} = \tau_{gi\alpha}$, if $\tau \subset \tau_\alpha$.*

Proof. If $\subset \tau_\alpha$, it implies to $\exists N \subset \tau_\alpha$, then $N \notin \tau$, and hence $\tau_{i\alpha} = po(x)$, equivalence to all subsets of $i\alpha$-closed set, since $\overline{\tau_{i\alpha}} \subseteq \overline{\tau_{gi\alpha}}$ by Corollary 9. Therefore $\tau_{i\alpha} = \tau_{gi\alpha}$. □

Theorem 24. *If $M \wedge N \subseteq \overline{\tau_{gi\alpha}}$, then $M \cup N \subseteq \overline{\tau_{gi\alpha}}$.*
(X, τ).

Proof. Let $M \wedge N \subseteq \overline{\tau_{gi\alpha}}$, and $U \subseteq \tau_{i\alpha}$, such that $M \subset U \wedge N \subset U$. Hence $Cl(M) \subseteq U, Cl(N) \subseteq U$. Since $\subset U \wedge N \subset U$, then $M \cup N \subseteq U$. Hence $Cl(M \cup N) = Cl(M) \cup Cl(N) \subseteq U$. Therefore $M \cup N$ is $gi\alpha$-closed set in τ. □

Theorem 25. *If $M \wedge N \subseteq \overline{\tau_{gi\alpha}}$, then $M \cap N \subset \overline{\tau_{gi\alpha}}$.*

Proof. Same the proof of above Theorem 24. □

Some expansion $gi\alpha$-mappings and their relationship

Definition 26. *The mapping $f : \hat{\tau} \to \hat{\sigma}$ is $gi\alpha$-continuous, if $f^{-1}(Q) \subseteq \tau_{gi\alpha}(\overline{\tau_{gi\alpha}}), \forall Q \subseteq \hat{\sigma}(\bar{\hat{\sigma}})$.*

Example 27. *Let $N = L\{a, r, h\}$, $\tau = \{\varnothing, \{r\}, N\}$, $\sigma = \{\varnothing, \{a\}, N\}$, $gi\alpha(N) = \{\varnothing, \{a\}, \{r\}, \{h\}, \{a, r\}, \{a, h\}, \{r, h\}, N\}$. It's obvious, the identity mapping $f : (N, \tau) \to (L, \sigma)$ is $gi\alpha$-continuous.*

Theorem 28. *If f is sg-continuous mapping, then f is $gi\alpha$-continuous mapping.*

Proof. Let $f : \hat{\tau} \to \hat{\sigma}$ be an sg-continuous mapping, $\forall V \subseteq \hat{\sigma}$. Since, f is sg-continuous, then $f^{-1}(V) \subseteq \tau_{gi\alpha}$. Since $\tau_{sg} \subseteq \tau_{gi\alpha}$ Theorem 15, then $f^{-1}(Q) \subseteq \tau_{gi\alpha}$. Therefore, f is an $gi\alpha$-continuous. □

Theorem 29. *Every $g\alpha$-continuous mapping is $gi\alpha$-continuous mapping.*

Proof. Same the prof above theorem by using Theorem 28. □

Theorem 30. *Every $r - g$-continuous mapping is $gi\alpha$-continuous mapping.*

Proof. Let $f : \hat{\tau} \to \hat{\sigma}$ be a $r - g$-continuous mapping, then $\forall V \subseteq \hat{\tau}$. Since f is $r - g$-continuous, then $f^{-1}(Q) \subseteq \tau_{rg}$. It implies $f^{-1}(Q) \subseteq \tau_{gi\alpha}$ since $\tau_{sg} \subset \tau_{gi\alpha}$ by Theorem 20. Therefore, f is an $gi\alpha$-continuous. □

Theorem 31. *Every $i\alpha$-continuous mapping is $gi\alpha$-continuous mapping.*

Proof. Same the prooof of the above Theorem 30 by using Theorem 12. □

The converse of above theorems is not ture generally.

Example 32. *Let* $H = \{1, 3, 5\}$, $K = \{2, 4, 6\}$, $\tau = \{\varnothing, \{1, 5\}, H\}$, $\sigma = \{\varnothing, \{4\}, K\}$,

$$gi\alpha(H) = \{\varnothing, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, H\},$$

$$i\alpha(H) = \{\varnothing, \{1\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, H\},$$

the map $f : (H, \tau) \to (K, \sigma)$ *is define as:* $f(1) = 6, f(3) = 4, f(5) = 2$. *Here is* f *is* $gi\alpha$*-continuous but is not* $i\alpha$ *(resp.* $sg, g\alpha$*, and* rg*)-continuous mappings, because* $f^{-1}\{4\} = \{3\}$ *is only* $gi\alpha$*-open set.*

Definition 33. *The mapping* $f : \hat{\tau} \to \hat{\sigma}$ *is* $gi\alpha$*-irresolute,* $f^{-1}(Q) \subseteq \hat{\tau}_{gi\alpha}, \forall Q \subseteq \hat{\sigma}_{gi\alpha}$.

Example 34. *Let* $Q = W = \{s, r, t\}$, $\sigma = \{\varnothing, \{t\}, Q\}$, $\tau = \{\varnothing, \{t\}, W\}$,
$gi\alpha(Q) = \{\varnothing, \{s\}, \{r\}, \{t\}, \{s, r\}, \{s, t\}, \{r, t\}, Q\}$,
$gi\alpha(W) = \{\varnothing, \{s\}, \{r\}, \{t\}, \{s, r\}, \{s, t\}, \{r, t\}, W\}$, $f : (Q, \tau) \to (W, \sigma)$ *is define as:* $f(r) = s, f(s) = t, f(t) = r$. *It is clear the mapping is* $gi\alpha$*-irresolute mapping.*

Theorem 35. *Every* α*(resp.* rg*, and* sg*)-irresolute function is* $gi\alpha$*-irresolute mapping.*

Proof. Let $f : \tau \to \sigma$, be an α-irresolute, and H any α (resp. rg, and sg)-open subset of σ, hence H is $gi\alpha$-open by Theorem 20 and Theorem 15. Since f is an α-irresolute, then $f^{-1}(H)$ is α-open in τ, and Since every α(resp. rg, and sg)-open set is $gi\alpha$-open set. Therefore f is $gi\alpha$-irresolute mapping. □

Theorem 36. *Every* $gi\alpha$*-irresolute is* $gi\alpha$*-continuous mapping.*

Proof. Clear form Definition 33. □

Theorem 37. *If* $f : \hat{\tau} \to \hat{\sigma}$ *is a* $gi\alpha$*-irresolute, and* $g : \hat{\sigma} \to \hat{Z}$ *mappings is also* $gi\alpha$*-irresolute, then* $f \circ g$ *is a* $gi\alpha$*-irresolute.*

Proof. Let R is $gi\alpha$-open set in $\hat{Z}$, since $g : \hat{\sigma} \to \hat{Z}$ is $gi\alpha$-irresolute , then $g^{-1}(R)$ is $gi\alpha$-open set in σ, since $f : \hat{\tau} \to \hat{\sigma}$ is a $gi\alpha$-irresolute, it imply $f^{-1}(g^{-1}(R))$ is $gi\alpha$-open set in τ. Therefore $f \circ g$ is a $gi\alpha$-irresolute. □

Corollary 38. *If* $g : \sigma \to \varphi$ *is* $gi\alpha$*-irresolute, and* $f : \tau \to \sigma$ *is* $gi\alpha$*-continuous mapping , prove* $f \circ g$ *is* $gi\alpha$*-irresolute.*

Proof. Same as the proof above Theorem 37. □

Definition 39. *The mapping* $f : \hat{\tau} \to \hat{\sigma}$ *is* $gi\alpha$*-open if,* $f(V) \subset \hat{\sigma}_{gi\alpha}, \forall V \subset \hat{\tau}_{gi\alpha}$.

Example 40. *Let* $\beta = \{5, 7, 9\}$, $\delta = \{4, 6, 8\}$, $\tau = \{\varnothing, \{9\}, \{7, 9\}, \beta\}$, $\sigma = \{\varnothing, \{8\}, \delta\}$, $gi\alpha(\delta) = \{\varnothing, \{4\}, \{6\}, \{8\}, \{4, 6\}, \{4, 8\}, \{6, 8\}, \delta\}$, $f : \hat{\tau} \to \hat{\sigma}$ *is define as:* $f(5) = 8, f(7) = 4, f(9) = 6$. *The mapping is* $gi\alpha$*-open mapping.*

Theorem 41. *Every* $i\alpha$*-open mapping is* $gi\alpha$*-open.*

Proof. Let $f : \hat{\tau} \to \hat{\sigma}$ is an $i\alpha$-open mapping, and $A \subset \tau$ then $f(A)$ is an $i\alpha$-open in $\hat{\sigma}$. Since $\tau_{gi\alpha} \subseteq \tau_{gi\alpha}$ Theorem 12, so $f(A)$ is $gi\alpha$-open. Therefore f is $gi\alpha$-open. □

Theorem 42. *Every* rg*-open mapping is* $gi\alpha$*-open.*

Proof. Same the proof the above Theorem 41 by using Theorem 20. □

The converse of Theorems 42, and 41 are not true.

Example 43. *Let* $B = \{1,2,3\}$, $C = \{4,5,6\}$, $\tau = \{\varnothing, \{2\}, B\}$, $\sigma = \{\varnothing, \{4,6\}, C\}$, $gi\alpha(C) = \{\varnothing, \{4\}, \{6\}, \{8\}, \{4,6\}, \{4,8\}, \{6,8\}, C\}$. *The identity mapping* $f : \hat{\tau} \to \hat{\sigma}$ *is* $gi\alpha$*-open mapping but it is not* $i\alpha$*-open and* rg*-open mapping, because* $f\{2\} = \{2\}$ *is not* $i\alpha$*-open and* rg*-open.*

Theorem 44. *If* $f : \hat{\tau} \to \hat{\sigma}$ *is open map, and* $g : \hat{\sigma} \to \hat{\varphi}$ *is* $gi\alpha$*-open map, then the composition* $f \circ g$ *is* $gi\alpha$*-open map.*

Proof. Let $H \subset \hat{\tau}$ since f is open map, then $f(H) \subset \hat{\sigma}$, and hence since g is $gi\alpha$-open map, then $g(f(H)) \subset \hat{\varphi}$ Therefore $f \circ g$ is $gi\alpha$-open map. ☐

Remark 5.

(i) If f *and* g *are* $gi\alpha$*-open map, then* $f \circ g$ *and* $g \circ f$ *are not necessary to be* $gi\alpha$*-open map.*

(ii) The proof of the above Theorems from (41 to 44) are true for closed mapping.

Theorem 45. *The bijection map* $f : X \to Y$, *the next sentences are equivalent.*

(a) $g^{-1} : \hat{\tau} \to \hat{\sigma}$ *is* $gi\alpha$*-continuous.*

(b) g *is* $gi\alpha$*-open map.*

(c) g *is* $gi\alpha$*-closed map.*

Proof.

$(a) \Rightarrow (b)$ Let $M \subset \tau$. Since f is $gi\alpha$-open mapping $g^{-1^{-1}}(M) = g(M)$ is is $gi\alpha$-open in (Y, σ). Therefore g is $gi\alpha$-open map.

$(b) \Rightarrow (c)$ Suppose $N \subset \overline{\hat{\tau}}$, then $\overline{N} \subset \hat{\tau}$. Since g is $gi\alpha$-open mapping $g(N^c)$ is $gi\alpha$-open mapping in $\hat{\sigma}$. Hence $g(N)$ is $gi\alpha$-closed in $\hat{\sigma}$. Therefore g is $gi\alpha$-closed mapping.

$(c) \Rightarrow (a)$ Let $\subset \hat{\tau}$. Since f is $gi\alpha$-closed mapping, then $f(R)$ is $gi\alpha$-closed in $\hat{\sigma}$, but $f(R) = f^{-1^{-1}}$. Hence $f^{-1} : \hat{\tau} \to \hat{\sigma}$ is $gi\alpha$- continuous. ☐

Definition 46. *If* $f : \hat{\tau} \to \hat{\sigma}$ *is* $gi\alpha$*-open,* $gi\alpha$*–continuous, and bijection, then* f *is* $gi\alpha$*–homeomorphism.*

Theorem 47. *If* f *is* rg*-homeomorphism, then* f *is* $gi\alpha$*–homeomorphism.*

Proof. Let $f : X \to Y$ is rg-homeomorphism. Since gr- continuous imply to $gi\alpha$-conti nuous by Theorem 30. Additionally since gr-open mapping imply to $gi\alpha$-open by Theorem 42. Therefore f is $gi\alpha$-homeomorphism. ☐

Theorem 48. *If* f *is* $i\alpha$*-homeomorphism, then* f *is* $gi\alpha$*–homeomorphism.*

Proof. Same the above Theorem 47 by using Theorem 31 and Theorem 41. ☐

Conclusion

The new $gi\alpha$-closed (open) set that mention in this paper investigated many properties for about the same type and the $gi\alpha$-mappings. We can expand about this new $gi\alpha$-set by related with other closed (open) sets and hence other mappings but we prefer to write in summary to put our readers in the main idea.

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The dissections and modular equations for Ramanujan-Selberg continued fraction

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First part in this report, we establish the 2-dissection and 4-dissection for Ramanujan-Selberg continued fraction and its reciprocal. Second part in this report, we give a short proof of a modular equation of Ramanujan-Selberg continued fraction given by Chan [Proc. Amer. Math. Soc. 137 (2009), 2849–2856].

2020 MSC: 11F27, 11P84, 11A55, 30B70

KEYWORDS: Ramanujan theta function, Jacobi triple product identity, Dissection, Reciprocal

For two indeterminates q and z with $|q| < 1$, the q-shifted factorial of infinite order is defined by

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k) = (1 - z)(1 - zq)\cdots(1 - zq^n)\cdots.$$

The Ramanujan's theta functions [5, p. 6 and p. 11] are defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{3}$$

where the product representation of the right side in (1)–(3) arises from Jacobi triple product identity, and the last equality in (3) is Euler's famous pentagonal number theorem.

Jacobi triple product identity is given by [5, p. 10, Theorem 1.3.3]

$$\begin{aligned}\sum_{n=-\infty}^{\infty} z^n q^{n^2} &= \prod_{n=1}^{\infty}(1 + zq^{2n-1})(1 + q^{2n-1}/z)(1 - q^{2n}) \qquad (4)\\ &= (-zq, -q/z, q^2; q^2)_\infty, \quad z \neq 0.\end{aligned}$$

The Ramanujan-Selberg continued fraction is defined by [3, p. 150] [17]

$$T(q) := \frac{1}{1+}\ \frac{q}{1+}\ \frac{q+q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^2+q^4}{1+}\cdots. \tag{5}$$

The dissections of Ramanujan-Selberg continued fraction.

Theorem 1. *The following* 2*-dissection of* $1/T(q)$ *holds*

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{(q,q^7,q^8;q^8)_\infty(q^6,q^{10};q^{16})_\infty}{(q^2;q^2)_\infty}, \tag{6}$$

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{(q^3,q^5,q^8;q^8)_\infty(q^2,q^{14};q^{16})_\infty}{(q^2;q^2)_\infty}. \tag{7}$$

Theorem 2. *The following* 4*-dissection of* $1/T(q)$ *holds*

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{(q^3,q^{13},q^{16};q^{16})_\infty(q^{10},q^{22};q^{32})_\infty}{(q;q)_\infty}, \tag{8}$$

$$\sum_{n=0}^{\infty} a_{4n+1}q^n = \frac{(q,q^{15},q^{16};q^{16})_\infty(q^{14},q^{18};q^{32})_\infty}{(q;q)_\infty}, \tag{9}$$

$$\sum_{n=0}^{\infty} a_{4n+2}q^n = -\frac{(q^5,q^{11},q^{16};q^{16})_\infty(q^6,q^{26};q^{32})_\infty}{(q;q)_\infty}, \tag{10}$$

$$\sum_{n=0}^{\infty} a_{4n+3}q^n = -q\frac{(q^7,q^9,q^{16};q^{16})_\infty(q^2,q^{30};q^{32})_\infty}{(q;q)_\infty}. \tag{11}$$

Theorem 3. *The following* 2*-dissection of* $T(q)$ *holds*

$$\sum_{n=0}^{\infty} b_{2n}q^n = \frac{(q,q^7,q^8;q^8)_\infty(q^6,q^{10};q^{16})_\infty}{(q^2;q^2)_\infty(q;q^2)^3_\infty}, \tag{12}$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^n = -\frac{(q^3,q^5,q^8;q^8)_\infty(q^2,q^{14};q^{16})_\infty}{(q^2;q^2)_\infty(q;q^2)^3_\infty}. \tag{13}$$

Theorem 4. *The following* 4*-dissection of* $(q^2;q^4)^3_\infty T(q)$ *holds*

$$\sum_{n=0}^{\infty} c_{4n}q^n = \frac{(q^3,q^{13},q^{16};q^{16})_\infty(q^{10},q^{22};q^{32})_\infty}{(q;q)_\infty}, \tag{14}$$

$$\sum_{n=0}^{\infty} c_{4n+1}q^n = -\frac{(q,q^{15},q^{16};q^{16})_\infty(q^{14},q^{18};q^{32})_\infty}{(q;q)_\infty}, \tag{15}$$

$$\sum_{n=0}^{\infty} c_{4n+2}q^n = -\frac{(q^5,q^{11},q^{16};q^{16})_\infty(q^6,q^{26};q^{32})_\infty}{(q;q)_\infty}, \tag{16}$$

$$\sum_{n=0}^{\infty} c_{4n+3}q^n = q\frac{(q^7,q^9,q^{16};q^{16})_\infty(q^2,q^{30};q^{32})_\infty}{(q;q)_\infty}. \tag{17}$$

Below we give a short proof of a modular equation of Ramanujan-Selberg continued fraction given by Chan [Proc. Amer. Math. Soc. 137, 2849–2856, 2009].

Theorem 5 (Jacobi's identity)**.** *For* $|q| < 1$

$$(q;q^2)^8_\infty + 16q(-q^2;q^2)^8_\infty = (-q;q^2)^8_\infty, \tag{18}$$

or, in standard notation,

$$\prod_{n\geq 1}(1-q^{2n-1})^8 + 16q\prod_{n\geq 1}(1+q^{2n})^8 = \prod_{n\geq 1}(1+q^{2n-1})^8.$$

Jacobi proved (18) in his famous work [Fundamenta nova theoriae functionum ellipticarum, Sumtibus Fratrum Borntraeger, Königsberg, 1829 (p. 90)], and describes this identity as "aequatio identica satis abstrusa" (a very well-concealed identity) [20, p. 470]. Equation (18) arises in many contexts and is often proved from relations satisfied by the classical Jacobi theta functions. For example, the proofs of (18) see the wonderful books by Whittaker and Watson [20], Borwein and Borwein [6, pp. 64-65] and Cooper [10, p. 175]; see also the elegant papers by Ewell [11], Chu [9, p. 66] and Chan [7].

Chan [7] gave a new proof of Jacobi's identity based on a key and fundamental ingredient, i.e., the following elegant modular equation of Ramanujan-Selberg continued fraction.

Theorem 6 (Chan's modular equation)**.** *Let* $x(q) = \dfrac{1}{2T^2(q)}$. *Then*

$$x(q) - \frac{q^{\frac{1}{2}}}{x(q)} = \frac{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty^4}{2(q^2; q^2)_\infty^4}, \tag{19}$$

or, equivalently that

$$\frac{1}{T^2(q)} - 4q^{\frac{1}{2}} T^2(q) = \frac{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty^4}{(q^2; q^2)_\infty^4}.$$

We below provide a very short proof of Chan's modular equation (19) (see [21]).

Proof of Theorem 6 First, it is easy to show that [4, p. 37, (22.3)]

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty}, \tag{20}$$

which is a very famous theorem of Euler. Next, by (1), (2) and (20), we rewrite the $T(q)$ with Ramanujan's theta function $\varphi(q)$ and $\psi(q)$ as

$$T(q) = \frac{(-q^2; q^2)_\infty}{(-q; q^2)_\infty} = \frac{\psi(q)}{\varphi(q)}. \tag{21}$$

It is known, the following identities are due to Jacobi [4, Entry 25, p. 40]:

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{22}$$

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4), \tag{23}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2). \tag{24}$$

It follows that, we obtain, from (21)–(24),

$$\begin{aligned} x(q) - \frac{q^{\frac{1}{2}}}{x(q)} &= \frac{1}{2T^2(q)} - 2q^{\frac{1}{2}} T^2(q) \\ &= \frac{\varphi(q)}{2\psi(q^2)} - 2q^{\frac{1}{2}} \frac{\psi(q^2)}{\varphi(q)} \end{aligned}$$

by (21) and (22)

$$=\frac{\varphi^2(q)-4q^{\frac{1}{2}}\psi^2(q^2)}{2\varphi(q)\psi(q^2)}$$

$$=\frac{\varphi^2(q^{\frac{1}{2}})+\varphi^2(-q^{\frac{1}{2}})-\left[\varphi^2(q^{\frac{1}{2}})-\varphi^2(-q^{\frac{1}{2}})\right]}{4\psi^2(q)}$$

by (22) again, (23) and (24)

$$=\frac{\varphi^2(-q^{\frac{1}{2}})}{2\psi^2(q)}$$

$$=\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty^2(q^{\frac{1}{2}};q)_\infty^2(q;q^2)_\infty^2}{2(q^2;q^2)_\infty^2}$$

$$=\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty^4}{2(q^2;q^2)_\infty^4}.$$

This completes the proof.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Nanomaterial design for engineering applications

Rukan Genc Alturk

Nanodesign covers the concepts of engineering and manipulating materials at the nanoscale to create novel, and functional structures and devices with unique properties and applications, which involves the intricate interplay of science, applied mathematics, machine learning, and engineering principles [2, 3]. By harnessing the potential of nanodesign, we can pave the way for groundbreaking advancements and transformative technologies in the future [1]. This presentation aims to discuss the application focused design of nanomaterials, showcasing their exceptional properties and exploring how they can enhance the performance of engineering systems. While nanodesign offers tremendous opportunities, it also presents challenges that need to be addressed. The presentation will examine these challenges and explore potential solutions. Safety and responsible use of nanomaterials will be a key focus, ensuring that nanodesign is pursued in order to mimize the potential risks, and hazards. The manipulation of surface properties of nanomaterials for the application of whim, and machine learning based selection of the specific properties of nanomaterials will be discussed. The surface properties of nanomaterials can be manipulated through a variety of methods, including chemical modification, physical modification, and deposition. This can be used to improve the performance of nanomaterials in a variety of applications, such as drug delivery, tissue engineering, sensors, and catalysis. In summary, this presentation will provide a comprehensive overview of nanodesign from a multidisiplinary perspective by covering strategies for apliaction focused design of nanomaterials and their unique properties. It will explore how nanomaterials can enhance engineering systems in various industries, and which strategies should be followed in order to adress the specific requirements of the application.

2020 MSC: 74M05, 65D17, 00A69

KEYWORDS: Smart nanomaterials, Machine learning, Nanodesign

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Some relations for the Pidduck polynomials

Rahime Dere

In this work, we study the Pidduck polynomials belonging to the family of the Sheffer polynomials. We give some relations of these polynomials by using by using the methods of the umbral calculus.

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KEYWORDS: Umbral algebra, Linear operators, Special polynomials

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An application of Hermite interpolation in explicit equation of algebraic curves

Ryotaro Okazaki

In 2013, five mathematicians met in Beijing for a research of algebraic curves. They were

J. W. Hoffman, D. Liang, Z. Liang, Y. Sakai, H. Wang.

The research was on finding

an explicit polynomial equation for the generic curves of genus 3 curves with the endomorphism ring containing cubic order of $\mathbb{Q}\left(\cos \frac{2\pi}{7}\right)$.

They identified the nature of of the complex algebraic curve Y of target class. Polynomials

polynomial:	$a(x)$	$b(x)$	$c(x)$	$s(x)$
degree:	7	4	2	6

satisfying

$$a(x)^2 - b(x)^2 s(x) = c(x)^7.$$

must be associated with the curve Y. The system

$$y^2 = s(x), \quad w^7 = a(x) + b(x)y$$

of equations defines a genus 8 curve. The product

$$\prod_{i=0}^{6} \left(v - \zeta^i w - \frac{c(x)}{\zeta^i w} \right)$$

expands in a polynomial $q(x, v)$ of x and v. More technically, the coefficients of the expansion are written in terms of coefficients of $a(x)$, $b(x)$, $c(x)$ and $s(x)$. Our curve Y is defined by

$$q(x, v) = 0.$$

The last blow would be a solution to the technical problem of finding the polynomials $a(x)$, $b(x)$, $c(x)$, $s(x)$. Then, the polynomial q should be found by a known method.

At this moment, I had the honor of participating in this ambitious research.

I introduced my knowledge of applied mathematics, technically Hermite Interpolation, in a solution of the equation in polynomials. I would like to survey our last blow.

2020 MSC: 14H10, 14Q05, 41A05

Keywords: Family (moduli) of algebraic curves, Computational aspects of algebraic curves, Interpolation in approximation theory (Specifically, Hermite interpolation)

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Some formula by convolution sums and the inverse divisor functions for coprime conditions

Soungdouk Lee [1] *and Daeyeoul Kim* [*2]

The value of this convolution sum first appeared in a letter from Besge to Loiuville in 1862. We obtain Besge's formula as a simple application of Liouville's identity. Glaisher extended Besge's formula by replacing $\sigma(n)$ in the convolution sum by other sums over the divisors of n.
We will find some formula's for convolution sum by Dirichlet convolution and the inverse functions of divisor functions.

2020 MSC: 11-XX

KEYWORDS: Dirichlet convolution, Restricted divisor functions

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Charlier polynomial solutions of Lane-Emden differential equations

Suayip Yuzbasi *1 *and Simge Yilmaz* 2

The main purpose of this study is to procure an approximate solution method for Lane-Emden type equations.This method converts Lane-Emden equations to matrix equations using collocation points and by the Charlier polynomials. These matrix equations correspond to linear differential equations. Unknown in these equations are the coefficients of the Charlier polynomial. Further examples are included to ensure effectiveness and applicability. The obtained results and error are compared with other methods that previously applied in the literature. Hence accuracy and reliability are documented. All the numerical computations have been performed on the program matlab R2023a.

2020 MSC: 65L10, 65L11, 65L60, 65L80, 65D05

KEYWORDS: Lane-Emden Equations, Charlier polynomials, Approximation method, Collocation method, Collocation points

Introduction

Lane–Emden equations are singular initial value problems relating to second-order ordinary differential equations(ODEs), which have been used to model several phenomena in mathematical physics and astrophysics such as thermal explosions [7], stellar structure [8], the thermal behaviour of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [4]. Lane-Emden equations have been solved by many methods in the literature, such as the bessel collocation method [9, 11], B-spline method [1], the pell-lucas collocation method [14]. In this study we are also going to use collocation method but based on Charlier polynomials on Lane-Emden equations in type

$$y''(x) + \frac{\alpha}{x}y'(x) + p(x)y(x) = g(x), \quad 0 \leq x \leq b \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{1} a_{jk}y^{(k)}(0) + b_{jk}y^{(k)}(b) = \lambda_j, \quad j = 0, 1 \tag{2}$$

where $y^{(0)}(x) = y(x)$ is an unknown function, the known functions $p(x)$ and $g(x)$ are defined on interval $0 < a \leq x \leq b$, and α, a_{jk}, b_{jk} and λ_j are real or complex constants. Our goal is to find the approximate solution of (1) stated in the truncated Charlier series form

$$y(x) = \sum_{n=0}^{N} a_n \mathbf{C_n}(x) \tag{3}$$

so that a_n, $n = 0, 1, 2, ..., N$ are unknown Charlier coefficients. We can choose N as any positive integer such that $N \geq 2$, and $C_n(x)$, $n = 0, 1, 2, ..., N$ are Charlier polynomials. Charlier polynomials can be given by the generating function as [2, 5]

$$\sum_{n=0}^{\infty} \mathbf{C_n}(x, \alpha) \frac{z^n}{n!} = \frac{(1+z)^x}{e^{az}} \tag{4}$$

and their explicit formulation can also be of the form [2, 5]

$$\mathbf{C_n}(x, \alpha) = \sum_{k=0}^{n} k! \binom{n}{k} \binom{x}{k} (-\alpha)^{n-k}. \tag{5}$$

Fundamental matrix relations

We can write Charlier polynomials as follows

$$\mathbf{C^T}(x) = \mathbf{L}\mathbf{X^T}(x) \iff \mathbf{C}(x) = \mathbf{X}(x)\mathbf{L^T} \tag{6}$$

where $\mathbf{C}(\mathbf{x}, \alpha) = \begin{bmatrix} \mathbf{C_0}(x, \alpha) & \mathbf{C_1}(x, \alpha) & \dots & \mathbf{C_N}(x, \alpha) \end{bmatrix}$, $\mathbf{X}(x) = \begin{bmatrix} 1 & x & \dots & x^N \end{bmatrix}$, and [3]

$$\mathbf{L^T}(\alpha) = \begin{bmatrix} l_{00}(\alpha) & 0 & 0 & ... & 0 \\ l_{10}(\alpha) & l_{11}(\alpha) & 0 & \dots & 0 \\ l_{20}(\alpha) & l_{21}(\alpha) & l_{22}(\alpha) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{N0}(\alpha) & l_{N1}(\alpha) & l_{N2}(\alpha) & \dots & l_{NN}(\alpha) \end{bmatrix}$$

such that [3]

$$l_{mn}(\alpha) = \begin{cases} n!\binom{m}{n}(-\alpha)^{m-n}, & m \geq n \\ 0, & m < n \end{cases}, \quad m, n = 0, 1, ..., N. \tag{7}$$

We consider the solution $y(x)$ and it's derivatives $y'(x)$ and $y''(x)$ defined by a truncated Charlier series (3). Then we can write (3) in the matrix form

$$y(x) = \mathbf{C}(x)\mathbf{A}, \quad \mathbf{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix} \tag{8}$$

or from relations (6), (8), we find the following matrix relation

$$y(x) = \mathbf{X}(x)\mathbf{L}^T\mathbf{A}. \tag{9}$$

At the same time, the derivatives $\mathbf{X'}(x)$ and $\mathbf{X''}(x)$ can be written interms of $\mathbf{X}(x)$ as

$$\mathbf{X'}(x) = \mathbf{X}(x)\mathbf{B^T} \; and \quad \mathbf{X''}(x) = \mathbf{X}(x)(\mathbf{B^T})^2 \tag{10}$$

where

$$\mathbf{B^T} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

By using equations (6) and (10), we obtain matrix forms

$$\mathbf{C'}(x) = \mathbf{X'}(x)\mathbf{L^T} = \mathbf{X}(x)\mathbf{B^T}\mathbf{L^T} \text{ and } \mathbf{C''}(x) = \mathbf{X''}(x)\mathbf{L^T} = \mathbf{X}(x)(\mathbf{B^T})^2\mathbf{L^T} \tag{11}$$

and through the equations (8) and (11) we have recurrence relations

$$y'(x) = \mathbf{C'}(x)\mathbf{A} = \mathbf{X}(x)\mathbf{B^T}\mathbf{L^T}\mathbf{A} \text{ and } y''(x) = \mathbf{C''}(x)\mathbf{A} = \mathbf{X}(x)(\mathbf{B^T})^2\mathbf{L^T}\mathbf{A}. \tag{12}$$

Method of solution

Now we can build the fundamental matrix equation corresponding to Eq. (1).To do this, we substitute the matrix relations (9) and (12) into (1) and obtain the matrix equation

$$\mathbf{X}(x)(\mathbf{B^T})^2\mathbf{L^T}\mathbf{A} + \frac{\alpha}{x}\mathbf{X}(x)\mathbf{B^T}\mathbf{L^T}\mathbf{A} + p(x)\mathbf{X}(x)\mathbf{L^T}\mathbf{A} = g(x). \tag{13}$$

To use in (13) we determine collocation points by [10, 12, 13]

$$x_i = a + \frac{b-a}{N} i, \quad i = 0, 1, ..., N, \quad 0 < a \leq x \leq b, \tag{14}$$

the system of matrix equations turn into

$$\mathbf{X}(x_i)(\mathbf{B^T})^2\mathbf{L^T}\mathbf{A} + \frac{\alpha}{x_i}\mathbf{X}(x_i)\mathbf{B^T}\mathbf{L^T}\mathbf{A} + p(x_i)\mathbf{X}(x_i)\mathbf{L^T}\mathbf{A} = g(x_i). \tag{15}$$

or briefly fundamental matrix equation is

$$\left\{\mathbf{X}(\mathbf{B^T})^2\mathbf{L^T} + \mathbf{S}\mathbf{X}\mathbf{B^T}\mathbf{L^T} + \mathbf{P}\mathbf{X}\mathbf{L^T}\right\}\mathbf{A} = \mathbf{G} \tag{16}$$

where

$$\mathbf{S} = \begin{bmatrix} \frac{\alpha}{x_0} & 0 & \dots & 0 \\ 0 & \frac{\alpha}{x_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\alpha}{x_N} \end{bmatrix}, \mathbf{P} = \begin{bmatrix} p(x_0) & 0 & \dots & 0 \\ 0 & p(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p(x_N) \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix} \; and \;\; \mathbf{X} = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & (x_0)^2 & \dots & (x_0)^N \\ 1 & x_1 & (x_1)^2 & \dots & (x_1)^N \\ 1 & x_2 & (x_2)^2 & \dots & (x_2)^N \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_N & (x_N)^2 & \dots & (x_N)^N \end{bmatrix}.$$

Therefore the fundamental matrix equation (16) corresponds to (13). This also can be written in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G} or \quad [\mathbf{W};\mathbf{G}]; \mathbf{W} = \mathbf{X}(\mathbf{B^T})^2\mathbf{L^T} + \mathbf{S}\mathbf{X}\mathbf{B^T}\mathbf{L^T} + \mathbf{P}\mathbf{X}\mathbf{L^T}. \tag{17}$$

Here, Eq. (17) corresponds to a system of $(N+1)$ linear algebraic equations with unknown Charlier coefficients $a_0, a_1, ..., a_N$. Through Eq. (9), the conditions (2) can be converted to matrix form as

$$\sum_{k=0}^{1}[a_{jk}\mathbf{X}(0) + b_{jk}\mathbf{X}(b)](\mathbf{B^T})^k\mathbf{L^T}\mathbf{A} = [\lambda_j], \quad j = 0, 1 \tag{18}$$

or we can express 18 briefly in the form

$$\mathbf{U_j}\mathbf{A} = [\lambda_j] \quad or \quad [\mathbf{U_j} : \lambda_j]; \quad j = 0, 1 \tag{19}$$

where

$$\mathbf{U_j} = \sum_{k=0}^{1}[a_{jk}\mathbf{X}(0) + b_{jk}\mathbf{X}(b)](\mathbf{B^T})^k\mathbf{L^T} = \begin{bmatrix} u_{j0} & u_{j1} & \dots & u_{jN} \end{bmatrix}, \quad j = 0, 1. \tag{20}$$

Then we replace the row matrices (19) by two rows of the matrix (17) and thus, we have the new augmented matrix

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}}. \tag{21}$$

For convenience, if last two rows of the matrix (17) are replaced, the new augmented matrix of the above system turns to:

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \dots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{(N-2)0} & w_{(N-2)1} & w_{(N-2)2} & \dots & w_{(N-2)N} & ; & g(x_{(N-2)}) \\ u_{00} & u_{01} & u_{02} & \dots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & u_{12} & \dots & u_{1N} & ; & \lambda_1 \end{bmatrix}. \tag{22}$$

If rank $\tilde{\mathbf{W}} = rank[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = N+1$, then we can write

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-\mathbf{1}}\tilde{\mathbf{G}}. \tag{23}$$

The unknown Charlier coefficients matrix $\mathbf{A}(\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix}^T)$ is determined by solving this linear system. $a_0, a_1, ..., a_N$ are substituted in Eq. (3). Thus we obtain the Charlier polynomial solution

$$y(x) = \sum_{n=0}^{N} a_n \mathbf{C_n}(x). \tag{24}$$

Numerical examples

In this section two numerical examples will be examined. All the calculations will be performed on . To demonstrate accuracy and effectiveness, the values of the exact solution $y(x)$, the approximate solution $y_N(x)$ and the absolute error function $e_N(x) = |y(x) - y_N(x)|$ in Tables and Figures.
The point $x = 0$ is the singular point of the Lane-Emden equations. In other words, at $x = 0$, the equation is undefined. Therefore, in our examples, we will start the collocation points from a point very close to 0, not from $x = 0$.

Example 1 (*cf.* [6]). *Firstly we will consider Lane-Emden equation given by*

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 6 + 12x + x^2 + x^3, \quad 0 \le x \le 1 \tag{25}$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 0 \tag{26}$$

and the approximate solution $y(x)$ by truncated Charlier series

$$y_3(x) = \sum_{n=0}^{3} a_n C_n(x) \tag{27}$$

where $N = 3, \alpha = 2, p(x) = 1$ and $g(x) = 6 + 12x + x^2 + x^3$. Hence, the collocation points (14) for $a = 0.00001, , b = 1$ and $N = 3$ are calculated as

$$x_0 = \frac{1}{10000}, \quad x_1 = \frac{1667}{5000}, \quad x_2 = \frac{6667}{10000}, \quad x_3 = 1 \tag{28}$$

and from Eq. (16), the fundamental matrix equation of the problem is

$$\left\{\mathbf{X}(\mathbf{B^T})^2\mathbf{L^T} + \mathbf{SXB^TL^T} + \mathbf{PXL^T}\right\}\mathbf{A} = \mathbf{G} \tag{29}$$

where

$$\mathbf{S} = \begin{bmatrix} 200000 & 0 & 0 & 0 \\ 0 & 49997/8333 & 0 & 0 \\ 0 & 0 & 200000/66667 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B^T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{L^T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 3 & -6 & 6 \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} 49999/8333 \\ 4313/425 \\ 11203/760 \\ 20 \end{bmatrix}.$$

The augmented matrix for this fundamental matrix is

$$[\mathbf{W};\mathbf{G}] = \begin{bmatrix} -399975 & 199976 & 8327/694 & 9/12500 & ; & 49999/8333 \\ -5153/264 & 11800/389 & -4057/112 & 5547/229 & ; & 4313/425 \\ -5725/122 & 16883/218 & -55207/637 & 20857/419 & ; & 11203/760 \\ -81 & 131 & -142 & 78 & ; & 20 \end{bmatrix}.$$

The matrix forms for the initial conditions from Eq. (19) are

$$\begin{bmatrix}\mathbf{U_0} & ; & \lambda_0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & ; & 0\end{bmatrix}$$

and

$$\begin{bmatrix}\mathbf{U_1} & ; & \lambda_1\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0 & 0 & ; & 0\end{bmatrix}.$$

From (23), the new augmented matrix based on the conditions is obtained as

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = \begin{bmatrix} -399975 & 199976 & 8327/694 & 9/12500 & ; & 49999/8333 \\ -5153/264 & 11800/389 & -4057/112 & 5547/229 & ; & 4313/425 \\ 1 & 0 & 0 & 0 & ; & 0 \\ -2 & 1 & 0 & 0 & ; & 0 \end{bmatrix}.$$

Solving this system, the unknown Charlier coefficients matrix (23) is found as

$$\mathbf{A} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 7/6 \end{bmatrix}.$$

Therefore when we substitute the Charlier coefficients matrix into Eq. (8), we obtain the approximate solution $y_3(x) = x^2 + x^3$, which is the exact solution of (25).

Example 2 (*cf.* [1])**.** *Consider Lane-Emden problem*

$$y''(x) + \frac{1}{x}y'(x) = (\frac{8}{8-x^2})^2, \quad 0 \leq x \leq 1 \tag{30}$$

with the initial conditions

$$y(1) = 0, \quad y'(0) = 0$$

with the exact solution $y(x) = 2\log(\frac{7}{8-x^2})$ *so that* $\alpha = 1$, $p(x) = 0$ *and* $g(x) = (\frac{8}{8-x^2})^2$. *Here, the collocation points* (14) *are computed for* $a = 0.1$, $b = 1$ *and various* N *and from Eq.* (16), *the fundamental matrix equation of the problem is*

$$\left\{\mathbf{X}(\mathbf{B^T})^2\mathbf{L^T} + \mathbf{SXB^TL^T}\right\}\mathbf{A} = \mathbf{G}. \tag{31}$$

Thus, by applying the same procedure with the previous example, we obtain the approximate solutions by Charlier polynomials of the problem for $N = 7, 10, 12$, *respectively,*

$$\begin{gathered} y_7(x) = 0.0004283785541353x^7 + 0.00078257440896x^6 + 0.000297736329813x^5 + \\ 0.0155416217356x^4 + 0.00000943658121405x^3 + 0.249999999788x^2 - \\ 6.82787160144e - 15x - 0.267059747385, \\ y_{10}(x) = 0.0000311158668094x^{10} - 0.0000497477652648x^9 + 0.000182824495134x^8 - \\ 0.0000427495529324x^7 + 0.00132048554228x^6 - 0.00000482213774128x^5 + \\ 0.0156257162146x^4 - 0.000000476112277381x^3 + 0.250000000001x^2 - \\ 1.95399252334e - 14x - 0.267062775015, \\ y_{12}(x) = 0.00000471334521532x^{12} - 0.0000117703255772x^{11} + 0.000031432054853x^{10} - \\ 0.0000189124757191x^9 + 0.000134205782973x^8 - 0.00000522379588483x^7 + \\ 0.00130358743721x^6 - 0.000000279962909577x^5 + 0.0156250308749x^4 - \\ 0.0000000157839488701x^3 + 0.25x^2 + 5.40678612992e - 14x - 0.267062784991, \end{gathered}$$

The numerical result of absolute error function obtained by current method (CM) for $N = 7, 10, 12$ *and B-spline method* [1] *are compared in Table 1. The absolute error functions obtained by the current method for* $N = 7, 10, 12$ *and B-spline method [1] are shown in the Figure 1. It is seen from Table 1 and Figure 1 that the result obtained by the current method (CM) is better than that obtained by B-spline method.*

x_i	Absolute errors with B-spline method [1] for $h = 1/20$	CM, $e_7(x_i)$	CM, $e_{10}(x_i)$	$e_{12}(x_i)$
0.0	2.6000e-005	3.03786e-06	1.02343e-08	2.57733e-10
0.2	2.6000e-005	3.04714e-06	1.02189e-08	2.57444e-10
0.4	2.4000e-005	3.04875e-06	1.02142e-08	2.57313e-10
0.6	1.9000e-005	3.04889e-06	1.02118e-08	2.50808e-10
0.8	1.1000e-005	3.04656e-06	1.02234e-08	2.63807e-11
0.9	6.0000e-006	2.62967e-06	9.87415e-09	7.3597e-10

Table 1: Comparison of the absolute errors Example 4.2 to B-spline method

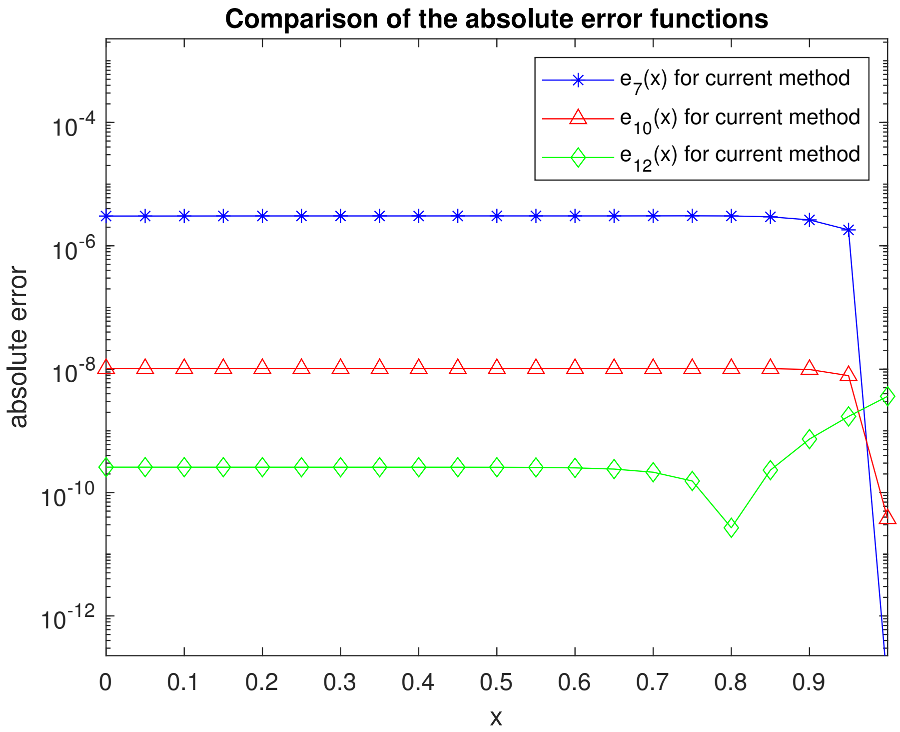

Figure 1: Comparison of the absolute error functions of Example 4.2.

Conclusion

We have developed a new approximate solution method to Lane-Emden type equations that can model phenomena related to many fields such as mathematical physics and astrophysics. This method has been proven to be reliable by demonstrating the accuracy of the data and applicability. It is also shown in the table that it is more capable than other methods. An advantage of this method is that it can be calculated very easily using MATLAB code. Since it is seen in the figure that the accuracy of the data increases when more collocation points are taken, a very close solution can be easily found by choosing N as high as possible on the code.

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A new approximation method for multi-pantograph type delay differential equations using Hermite polynomials

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In this article, a new approaching technique is presented to solve multi-pantograph type delay differential equations. Proposed new method is a collocation method based on integrating and Hermite polynomials. As the main idea of the method, the process starts by approaching the first derivative function in the equation in the form of truncated Hermite series. And after this approximating form is composed in the matrix form. Then, unknown function is acquired as a matrix form by integrating the approximating of the first derivative function. Using the approximation of the unknown function, the matrix forms of the proportionally delayed terms in the equation are obtained. In addition, necessary operational matrix forms are created for convenience in the method. By using these matrix forms and matrix operations, the problem is reduced to a system of algebraic linear equations. The method is explained with numerical applications. The results are compared with other methods known in the literature. In these comparisons, the effectiveness and reliability of the current method are clearly seen.

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KEYWORDS: Pantograph equation, Hermite polynomials, Collocation method

Introduction

Delay differential equations have an important place in modeling many problems in science. Pantograph equations are the most widely used and important types of delay differential equations, and these equations have an important position in many fields such as electrodynamics [6], physics [5], control systems, chemistry [3], electronic systems [9] and medicine [8]. Many methods are presented to solve multi-pantograph equations such as variational iteration method [14], spectral method [1], galerkin method [16], hybrid Taylor-Lucas collocation method [2], Laguerre approach method [15], Bessel method [17], Runge Kutta method [7], Hermite method [13]. This research goals to find an approximation method to solve multi-pantograph type delayed differential equations specified by [16]

$$u'(x) = f(x) + \alpha(x)u(x) + \sum_{i=1}^{M} \beta_i(x)u(q_i x), \quad a \leq x \leq b, \quad 0 < q_i < 1 \tag{1}$$

via initial condition $u_N(a) = u_a$. In addition, the method based on Hermite polynomials using the derivative relation is already available [13]. Our study differs in that it is constructed using the integral and therefore better results are obtained for the same N values than the other Hermite method. With this in mind, we create the new method and its advantages will be demonstrated in applications. An approximate solution specified in the the truncated series of the Hermite polynomials $u'(x)$ as follows

$$u'_N(x) = \sum_{\mu=0}^{N} a_\mu H_\mu(x). \tag{2}$$

Here, any $N \in \mathbb{Z}$ is chosen and $H_n(x)$, $n = 0, 1, 2, ..., N$ are the Hermite polynomials described by [13]

$$H_n(x) = \sum_{k=0}^{[|\mu/2|]} (-1)^k \frac{\mu!}{k!(\mu-2k)!} 2x^{\mu-2k}. \tag{3}$$

In the continuation of this study, the main matrix representations will be presented (in Section 2). The method will be created in section 3 by using the obtained matrix relations. Supporting tables and graphs will be given about the effectiveness of the method by giving applications in section 5. Finally, the results of the research will be discussed in the conclusion part of the 6th Section.

Fundamental matrix relations

The matrix representation of Hermite polynomials can be formed as

$$\mathbf{H}(x) = \mathbf{X}(x)\mathbf{C} \tag{4}$$

where,

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^N \end{bmatrix}, \quad \mathbf{H}(x) = \begin{bmatrix} H1(x) & H_2(x) & H_3(x) & \dots & H_N(x) \end{bmatrix}$$

and if N is odd,

$$\mathbf{C} = \begin{bmatrix} \frac{1!}{2^0 0!0!} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1!}{2!1!1!} & 0 & \dots & 0 & 0 \\ \frac{2!}{2^2 0!1!} & 0 & \frac{2!}{2^2 2!1!} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{N!}{2^N 0! \frac{N}{2}!} & 0 & \frac{N!}{2^N 2!(\frac{N}{2}-1)!} & \dots & 0 & \frac{N!}{2^N N!0!} \end{bmatrix}$$

and if N is even,

$$\mathbf{C} = \begin{bmatrix} \frac{0!}{2^0 0!0!} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1!}{2!1!1!} & 0 & \dots & 0 & 0 \\ \frac{2^1}{2^2 0!1!} & 0 & \frac{2!}{2^2 2!1!} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{N!}{2^N 2!(\frac{N}{2}-1)!} & 0 & \dots & 0 & \frac{N!}{2^N N!0!} \end{bmatrix}$$

Eq. (2) can be typeable in the matrix form as

$$u'_N(x) = \mathbf{X}(x)\mathbf{C}\mathbf{A} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_N \end{bmatrix}^T, \mathbf{X} = \begin{bmatrix} 1 & x & x^2 & \dots & x^N \end{bmatrix}.$$

On the other hand we can integrate both side of Eq. (5) from a to x, then we get

$$\int_a^x u_N'(x) = \int_a^x \mathbf{X}(x)\mathbf{C}\mathbf{A}\, dx \tag{6}$$

and when the condition $u_N(a) = u_a$ is used, we get

$$u_N(x) = u_a + [\tilde{\mathbf{X}}(x) - \tilde{\mathbf{X}}(a)]\mathbf{C}\mathbf{A}, \tag{7}$$

where

$$\tilde{\mathbf{X}} = \begin{bmatrix} x & \frac{x^2}{2} & \frac{x^3}{3} & \dots & \frac{x^{N+1}}{N+1} \end{bmatrix}.$$

Also we can define new matrix as

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{N+1}- \end{bmatrix},$$

so the matrix $\tilde{\mathbf{X}}$ can be typed in the form

$$\tilde{\mathbf{X}} = x\mathbf{X}(x)\mathbf{V}. \tag{8}$$

Eq. (8) is substituted in equation (7), we have

$$u_N(x) = u_a + [x\mathbf{X}(x)\mathbf{V} - a\mathbf{X}(a)\mathbf{V}]\mathbf{C}\mathbf{A}. \tag{9}$$

Substituting $x \to q_i x$ in the equation (9), we get

$$u_N(q_i x) = u_a + [q_i x\mathbf{X}(q_i x)\mathbf{V} - a\mathbf{X}(a)\mathbf{V}]\mathbf{C}\mathbf{A}. \tag{10}$$

Matrix $\mathbf{X}(q_i x)$ can be formed as

$$\mathbf{X}(q_i x) = \mathbf{X}(x)\mathbf{Q}(q_i), \tag{11}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q_i^1 & 0 & \dots & 0 \\ 0 & 0 & q_i^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q_i^N \end{bmatrix}.$$

We subtitute Eq. (11) in Eq. (10) and thus, we get

$$u_N(q_i x) = u_a + [q_i x\mathbf{X}(x)\mathbf{Q}(q)\mathbf{V} - a\mathbf{X}(a)\mathbf{V}]\mathbf{C}\mathbf{A}. \tag{12}$$

Method of solution

Firstly, let's put Eqs. (2), (9) and (12) into Eq. (1). Then we obtain

$$\mathbf{X}(x)\mathbf{C}\mathbf{A} = f(x) + \alpha(x)(u_a + [x\mathbf{X}(x)\mathbf{V} - a\mathbf{X}(a)\mathbf{V}]\mathbf{C}\mathbf{A}) +$$
$$\sum_{i=1}^{M} \beta_i(x)(u_a + [q_i x\mathbf{X}(x)\mathbf{Q}(q_i)\mathbf{V} - a\mathbf{X}(a)\mathbf{V}]\mathbf{C}\mathbf{A}). \tag{13}$$

Rearranging (13), we get

$$\left[\mathbf{X}(x)-\alpha(x)[x\mathbf{X}(x)-a\mathbf{X}(a)]\mathbf{V}-\sum_{i=1}^{M}\beta_i(x)[q_i x\mathbf{X}(x)\mathbf{Q}(q_i)-a\mathbf{X}(a)]\mathbf{V}\right]\mathbf{CA}=$$

$$f(x)+\alpha(x)u_a+\sum_{i=1}^{M}\beta_i(x)u_a. \tag{14}$$

The collocation points are defined by

$$x_j=a+\frac{b-a}{N}j, \qquad j=0,1,...,N.$$

Substituting these collocation points in Eq. (14), we obtain the system

$$\left[\mathbf{X}(x_j)-\alpha(x_j)[x_j\mathbf{X}(x_j)-a\mathbf{X}(a)]\mathbf{V}-\sum_{i=1}^{M}\beta_i(x_j)[q_i x_j\mathbf{X}(x_j)\mathbf{Q}(q_i)-a\mathbf{X}(a)]\mathbf{V}\right]\mathbf{CA}=$$

$$f(x_j)+\alpha(x_j)u_a+\sum_{i=1}^{M}\beta_i(x_j)u_a, \qquad j=0,1,...,N, \tag{15}$$

or briefly the fundamental matrix equation is

$$\left[\mathbf{X}-\mathbf{K}[\bar{\mathbf{X}}\mathbf{X}-a\mathbf{X}(a)]\mathbf{V}-\sum_{i=1}^{M}\mathbf{B}[q_i\bar{\mathbf{X}}\mathbf{X}\mathbf{Q}(q_i)-a\mathbf{X}(a)]\mathbf{V}\right]\mathbf{CA}=$$

$$\mathbf{F}+\bar{\mathbf{K}}u_a+\sum_{i=1}^{M}\bar{\mathbf{B}}u_a, \tag{16}$$

where

$$\mathbf{X}=\begin{bmatrix}\mathbf{X}(x_0)\\ \mathbf{X}(x_1)\\ \mathbf{X}(x_2)\\ \vdots\\ \mathbf{X}(x_N)\end{bmatrix}, \mathbf{K}=\begin{bmatrix}\alpha(x_0) & 0 & 0 & \dots & 0\\ 0 & \alpha(x_1) & 0 & \dots & 0\\ 0 & 0 & \alpha(x_2) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \alpha(x_N)\end{bmatrix}, \bar{\mathbf{K}}=\begin{bmatrix}\alpha(x_0)\\ \alpha(x_1)\\ \alpha(x_2)\\ \vdots\\ \alpha(x_N)\end{bmatrix},$$

$$\bar{\mathbf{X}}=\begin{bmatrix}x_0 & 0 & 0 & \dots & 0\\ 0 & x_1 & 0 & \dots & 0\\ 0 & 0 & x_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & x_N\end{bmatrix}, \mathbf{B}=\begin{bmatrix}\beta_i(x_0) & 0 & 0 & \dots & 0\\ 0 & \beta_i(x_1) & 0 & \dots & 0\\ 0 & 0 & \beta_i(x_2) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \beta_i(x_N)\end{bmatrix},$$

$$\bar{\mathbf{B}}=\begin{bmatrix}\beta_i(x_0)\\ \beta_i(x_1)\\ \beta_i(x_2)\\ \vdots\\ \beta_i(x_N)\end{bmatrix},\ \mathbf{F}=\begin{bmatrix}f(x_0)\\ f(x_1)\\ f(x_2)\\ \vdots\\ f(x_N)\end{bmatrix}.$$

So, the fundamental matrix equation suitabing to Eq. (1) can be typed as

$$[\mathbf{W}:\mathbf{G}] \tag{17}$$

where,

$$\mathbf{G} = \mathbf{F} + \bar{\mathbf{K}}u_a + \sum_{i=1}^{M} \bar{\mathbf{B}}(x)u_a \tag{18}$$

and

$$\mathbf{W} = \left[\mathbf{X} - \mathbf{K}[\bar{\mathbf{X}}\mathbf{X} - a\mathbf{X}(a)]\mathbf{V} - \sum_{i=1}^{M} \mathbf{B}[q_i\bar{\mathbf{X}}\mathbf{X}\mathbf{Q}(q_i) - a\mathbf{X}(a)]\mathbf{V}\right]\mathbf{C}. \tag{19}$$

If this linear algebraic system is solved, the matrix $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_N \end{bmatrix}$ is obtained. Then $\mathbf{A}$ is replaced in Eq. (9) and therefore we obtain the approximate solution

$$u_N(x) = u_a + [x\mathbf{X}(x)\mathbf{V} - a\mathbf{X}(a)\mathbf{V}]\mathbf{C}\mathbf{A}. \tag{20}$$

Applications

In this section, an application is discussed to demonstrate the reliability of the method. The obtained data are compared with different N values and methods in tables and graphs.

Example 1. *Let us next consider the second application equation as* [10, 12, 16]

$$u'(x) = \frac{u(x)}{2} + \frac{1}{2}e^{(x/2)}u(x/2), \qquad 0 \leq x \leq 1 \tag{21}$$

with initial conditions $u(0) = 1$*. For this equation* $\alpha = 1/2$*,* $\beta = (1/2)e^{x/2}$*,* $q = 1/2$ *and* $f(x) = 0$*. The solution that satisfies the equation of this equation is* $u(x) = e^x$*. The set of collocation points is calculated for* $N = 3$ *as*

$$\left\{ x_0 = 0, \qquad x_1 = \frac{1667}{5000}, \qquad x_3 = \frac{6667}{10000}, \qquad x_4 = 1 \right\}.$$

From this point of view, we can calculate the $\mathbf{W}$ *matrix as*

$$\mathbf{W} = \left[\mathbf{X} - \mathbf{K}[\bar{\mathbf{X}}\mathbf{X} - 0\mathbf{X}(0)]\mathbf{V} - \mathbf{B}[0.5\bar{\mathbf{X}}\mathbf{X}\mathbf{Q}(0.5) - 0\mathbf{X}(0)]\mathbf{V}\right]\mathbf{C},$$

$$\mathbf{G} = \mathbf{F} + \bar{\mathbf{K}} + \bar{\mathbf{B}}$$

where

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1667/5000 & (1667/5000)^2 & (1667/5000)^3 \\ 1 & 6667/10000 & (6667/10000)^2 & (6667/10000)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\bar{\mathbf{X}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 0 & 1667/5000 & 0 & 0 & \\ 0 & 0 & 6667/10000 & 0 & \\ 0 & 0 & 0 & & 1/2 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix},$$

$$\mathbf{Q}(0.2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (1/2) & 0 & 0 \\ 0 & 0 & (1/2)^2 & 0 \\ 0 & 0 & 0 & (1/2)^3 \end{bmatrix}, \bar{\mathbf{K}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \bar{\mathbf{B}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix}1/2&0&0&0\\0&1/2&0&0\\0&0&1/2&0\\0&0&0&1/2\end{bmatrix},\mathbf{G}=\begin{bmatrix}1\\433/397\\3276/2735\\984/743\end{bmatrix}.$$

From this point of view, we apply the method and gain the $\mathbf{W}$ *matrix as*

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}1&0&-2&0&;&1\\1167/1588&1033/1737&-589/559&-2638/803&;&433/397\\1761/4057&708/685&1007/1486&-1853/458&;&3276/2735\\261/2972&1466/1133&4017/1330&-357/412&;&984/743\end{bmatrix}.$$

If we solve the system and replace its solution (that is; the specified coefficients) in Eq. (20)*, we acquire an approximate solution for* $N=3$*. After solving the linear algebraic system, matrix* $\mathbf{A}$ *is found as*

$$\mathbf{A}=\begin{bmatrix}40/33&839/1170&2665/25127&209/5981\end{bmatrix}$$

and replace in Eq. (20)*, we get approximate solution for N=3 as*

$$u_3(x)=418/5981x^4+1779/12580x^3+922/1817x^2+x+1.$$

x	Present Method $N=7$	Taylor Method [11] $N=7$	Galerkin Method [16] $N=7$	Present Method $N=9$	Taylor Method [11] $N=9$	Legendre P. Method [4] $N=9$
0,0	0	0	0	0	0	0
0,2	3.8503e-10	2,5600e-11	6.2610e- 09	3.4617e-13	2.2200-14	4.9000e–13
0,4	4.5797e-10	3.3300e-09	1.0890e-08	4.3578e-13	2.2200e-10	3.3600e–12
0,6	5.3296e-10	7.7700e-08	1.0810e-08	5.3335e-13	2.2200e-09	2.5040e–11
0,8	6.4064e-10	4.3800e-07	1.4970e-08	6.6051e-13	1.3300e-08	4.5530e–11
1,0	1.1526e-09	2.1200e-06	1.7880e-08	1.1186e-12	5.0100e-07	4.8860e–11

Table 1: Comparising absolute errors of Eq. (21)

In Table 1, the results for different N values are given and compared with different methods. Exact solution and approximate solution for N=3 which is found by present method are compared in Figure 1. In Figure 2, the absolute error function of the present method is compared with different methods.

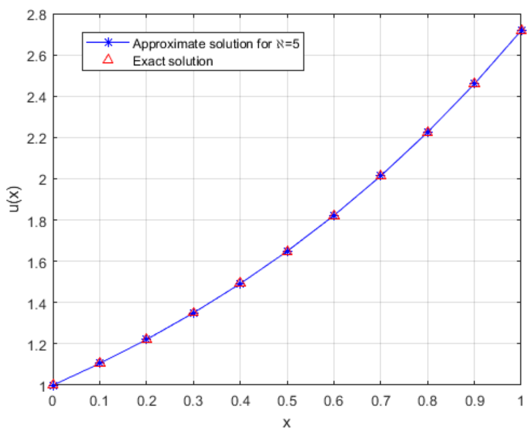

Figure 1: Comparising approximate solution $u_5(x)$ with exact solution $u(x)$ of Example 1

It turns out that our method slightly outperforms the Taylor method with the same N values, especially for larger values of x. Also, the error in our method does not change significantly over the $[0, 1]$ interval. As a result, if the solution is an exponential equation, the results obtained are remarkably close to the exact solution.

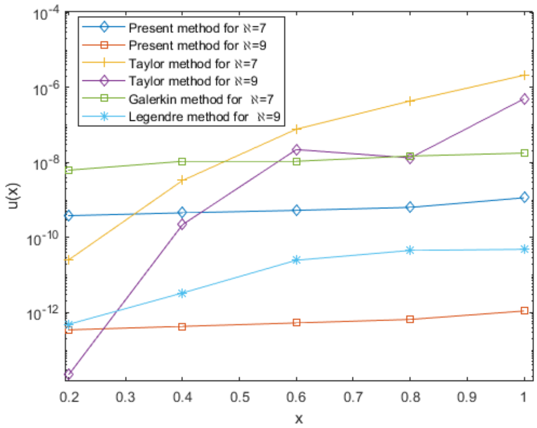

Figure 2: Graphics of the absolute error functions for diffirent methods and the current method of Example 1 (for different values of N)

Conclusion

In this research, a new approximation method is presented using Hermite polynomials to solve multi-pantograph type delay differential equations. In order to better understand the current method and determine its usability, the current method is explained and the applications that have been handled with different methods before are given in tables and graphics. Table 1 was created from the comparison made with different methods. The error values here are important to discuss the effectiveness of the method. When the known functions in the equation are extended to the Hermite series, it is seen that this method has a better outcome than the other polynomial methods.

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Ramanujan congruences for partitions functions associated to eta-quotients modulo prime powers

Sofiane Atmani [*1] *and Abdelmejid Bayad* [2]

In this talk, we investigate the generalized partition functions $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ given by their generating functions

$$\sum_{n=0}^{\infty} p_{[1^{\lambda_0} l^{k\lambda_1}]}(n) q^n = \frac{q^{\frac{\lambda_0 + l^k \lambda_1}{24}}}{\eta(\tau)^{\lambda_0} \eta(l^k \tau)^{\lambda_1}},$$

where $\eta(\tau)$ is the Dedekind eta function, l prime number, k positive integer and λ_0, λ_1 arbitrary integers. We prove Ramanujan's type congruences for the partition functions $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ modulo powers of $l \in \{2, 3, 5, 7, 11, 13, 17\}$ for any integers λ_0 and λ_1. Our results confirm and extend the previous ones of Atkin, Gordon, Wang, and others regarding congruences for various partition functions. For $l = 11$, we have made significant improvements to Gordon's and Wang's results.

Acknowledgments

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Order reduction mitigating predictor-corrector strategy for fractional differential equations

Yonghyeon Jeon[*1] *and Sunyoung Bu*[2]

We implement the predictor-corrector strategy by designing a new explicit formula based on Adams-Bashforth. Applying the explicit formula to obtain an enhanced predictor, which greatly mitigates order reduction. For the fractional differential equation, we demonstrate that the proposed strategy can augment the overall convergence com- pared to the traditional predictor technique. Theoretical consideration is also provided for the proof of convergence order, supported by er- ror analysis. Furthermore, we conduct numerical tests to validate the convergence order of the proposed scheme. As a result, the effective- ness of the third-order scheme and the significant mitigation of order reduction were achieved by using the enhanced predictor strategy.

2020 MSC: 34A08, 65L05, 65M12

Keywords: fractional differential equation, fractional Adams method, predictor-corrector scheme

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An example of predicting mass movements with mathematical models: Suluada (Antalya–Turkey)

Yasemin Leventeli

In the southwest of Antalya, there are many islands of various sizes, which are natural wonders; such as Ucadalar, Besadalar and Suluada. However, Suluada is in danger of disappearing in the long term due to mass movements. In this study, mass movements in the form of debris flow were detected on the island. Modeling them and determining the improvement-prevention methods are the essences of this study.

2020 MSC: 00A71, 92F05, 70G40

Keywords: Geology, Debris flow, Mathematical models, Suluada, Antalya

Introduction

Mass movements are natural phenomenon, and sometimes, they cause loss of life and property. In addition, there are also cases where natural and cultural heritages are damaged. Suluada, is an island called "Turkey's Maldives" and one of the important tourism centers (Figure 1), loses land due to debris flow and rockfall, which are types of mass movements, and the island faces the danger of extinction over time [12].

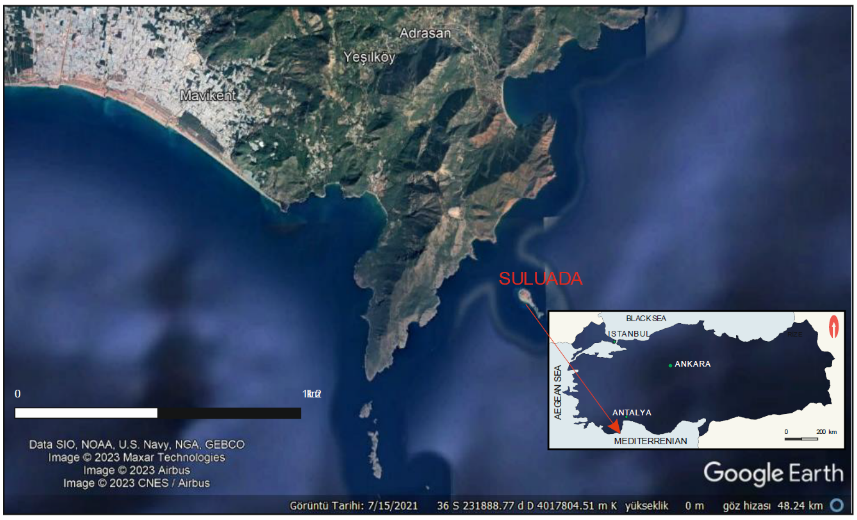

Figure 1: Location map of study area [3]

Many studies have been carried out on different subjects on the mainland, which forms the southwest of Antalya [7, 10, 11, 14, 15, 16]. However, no literature related to the study area was found.

In this study, the debris flow, which is common on the island, was tried to be modeled. Mathematical modeling of mass movements in different regions has been done by many researchers using different methods [1, 2, 4, 5, 8, 17, 18, 19].

Discussions

Mass movements are classified according to the type of motion and material [13]. There are many factors in the formation of mass movements; gravity, geological-hydrogeological- climatic conditions, earthquake, topography, vegetation, etc. In this study, primarily geological and topographic conditions were considered. Engineering features of geological units, which depend on lithological and structural properties of them, are very effective in the formation of its topography.

"Debris" is the accumulation of clay-silt-sand-gravel and blocks separated from the main rock as a result of mechanical weathering of the rock. These accumulations like alluvial fans in the study area. However, the movement has been termed "debris flow" because of its formation mechanism.

The island, which is located on the 1/25000 scale P24-c22 map, has been defined by the MTA [9] as a Mesozoic aged limestone. However, for the first time, the island was mapped as ophiolitic mélange, limestone, brecciated limestone, talus and beach deposits in this study. These debris flows are derived from Jurassic-Cretaceous aged limestones. They contain grains of different sizes, from clay-silt to coarse blocks (Figure 2).

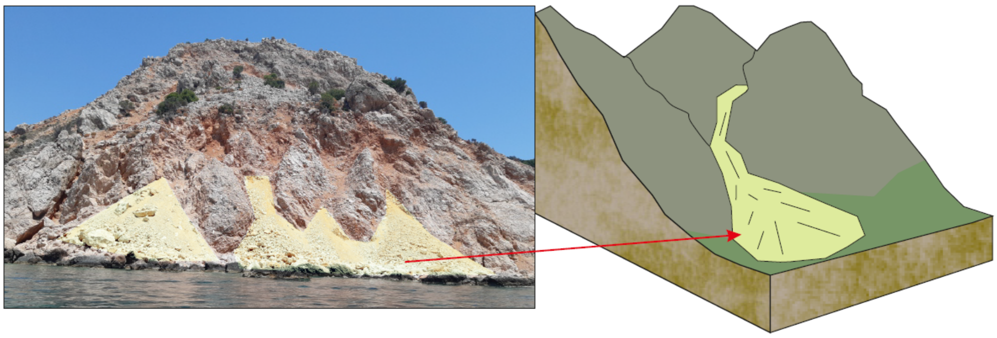

Figure 2: A photo of debris flows in the study area and their schematic representation

Debris-flow activity is strongly controlled by hydro-meteorological trigger conditions, which are expected to change [6]. In [6], changes in the frequency of critical trigger conditions for different trigger types (long-term precipitation, short-term storm, snowmelt, rain) were evaluated. It is thought that the results of this study may serve as a basis for the development of compliance strategies for future risk management. Therefore, the model called Tigger model will be applied for these debris flows in the study area, in the future.

The Tigger model is given in the following equation.

$$\Delta N = \left(\frac{\widehat{N}_F + 1}{\widehat{N}_R + 1} \right) x100$$

- the future period; $\widehat{N}_F$
- the reference period; $\widehat{N}_R$
- the mean annual change of trigger condition frequency; ΔN

Conclusions

Current and/or potential mass movements in Suluada cause land loss in the long term. With this study, the formation mechanisms of potential mass movements occurring on the island were revealed, the cause-effect relationship was examined. Besides that, mathematical model which is called Tigger model will be used for these purposes. The last study will be to determine the prevention methods.

Acknowledgments

This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Bijection between simple directed lattice paths and AND/OR tree structures

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This article studies the issues related to the combinatorial generation of simple directed lattice paths. To obtain new algorithms for combinatorial generation, it is proposed to apply the method based on the use of AND/OR trees. For several different types of simple directed lattice paths, recurrences are obtained that satisfy the requirements of the combinatorial generation method used. The corresponding AND/OR tree structures are also constructed, bijection rules between their variants and the considered lattice paths are defined, and algorithms for combinatorial generation are developed.

2020 MSC: 68R05, 05C38

KEYWORDS: Directed lattice path, Recurrence, AND/OR tree, Bijection

Introduction

Lattice paths are one of the main combinatorial objects widely used in discrete mathematics [1, 2]. A simple graphical representation of lattice paths allows us to see the specific shape of a combinatorial object. At the same time, it is possible to map a set of specific lattice paths into another set of more complex combinatorial objects.

A lattice path P is a sequence $P = (P_0, P_1, \ldots, P_l)$ of points P_i in the d-dimensional integer lattice (i.e., $P_i = (p_{i1}, p_{i2}, \ldots, p_{id})$, $p_{ij} \in \mathbb{Z}$, $P_i \in \mathbb{Z}^d$), where P_0 is the starting point and P_l is the end point. In this paper, we consider lattice paths in the plane, i.e. $d = 2$.

It is also required to specify a set $S = \{s_1, s_2, \ldots, s_k\}$ of possible steps in the lattice path, where each step s_i is a vector in the d-dimensional integer lattice (i.e., $s_i \in \mathbb{Z}^d$). Based on the set of possible steps, a lattice path $P = (P_0, P_1, \ldots, P_l)$ can be represented as a sequence of steps performed $P = (\overrightarrow{P_0P_1}, \ldots, \overrightarrow{P_{l-1}P_l})$, where $\overrightarrow{P_{i-1}P_i} \in S$. For the 2-dimensional case, we have $s_i = (a_i, b_i)$ where $a_i, b_i \in \mathbb{Z}$.

This paper is devoted to the enumeration and generation of the following subclass of lattice paths in the plane: simple directed lattice paths [3]. A directed lattice path is a lattice path in the plane where each possible step $s_i = (a_i, b_i)$ has $a_i > 0$. A simple directed lattice path is a directed lattice path where each possible step $s_i = (a_i, b_i)$ have $a_i = 1$. If a simple directed lattice path begins at the origin $P_0 = (0, 0)$ and consists of n steps of the form $(1, b_i)$, then it ends at $P_n = (n, m)$, $m \in \mathbb{Z}$. Figure 1 presents an example of a simple directed lattice path that begins at $(0, 0)$ and ends at $(n, m) = (9, 1)$.

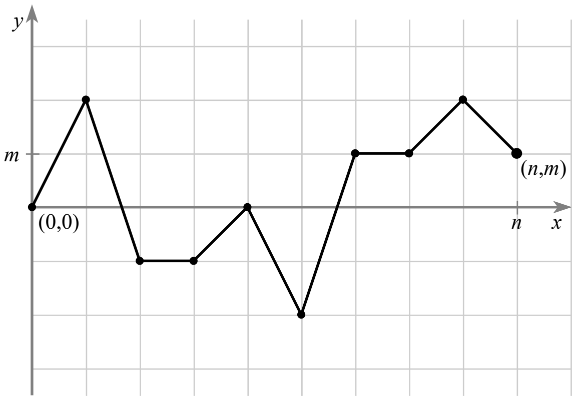

Figure 1: An example of a simple directed lattice path

To enumerate the considered simple directed lattice paths, recurrence relations will be used. To generate such lattice paths, the method based on the use of AND/OR trees will be applied [4, 5].

Main results

Let W_n^m denote the number of all simple directed lattice paths that begin at $(0,0)$ and end at (n,m). Next, we obtain a recurrence for W_n^m based on a graphical representation of the possible steps in the simple directed lattice paths.

Theorem 1. *The number of all simple directed lattice paths that begin at* $(0,0)$*, end at* (n,m) *and consist of steps* $\{(1,b_1),(1,b_2),\ldots,(1,b_k)\}$ *can be calculated using the following recurrence:*

$$W_n^m = \sum_{i=1}^{k} W_{n-1}^{m-b_i}, \tag{1}$$

where $W_0^0 = 1$ *and* $W_0^m = 0$ *for* $m \neq 0$.

To reduce the number of recursive calls, Equation (1) can be supplemented with the following initial conditions:

- $W_n^m = 0$ for $m > n \cdot \max b_i$;
- $W_n^m = 0$ for $m < n \cdot \min b_i$.

The obtained Equation (1) satisfies the requirements of the method based on AND/OR trees. Hence, it is possible to construct the corresponding AND/OR tree structure for W_n^m. Figure 2 presents an AND/OR tree structure for W_n^m.

The obtained AND/OR tree structure for W_n^m has the following initial conditions:

- if we get a node labeled W_0^0, then it is a leaf node;
- if we get a node labeled W_n^m and $W_n^m = 0$, then it is necessary to remove this node.

The obtained AND/OR tree structure does not contain AND nodes, therefore, each variant of this tree is a path from the root to a leaf.

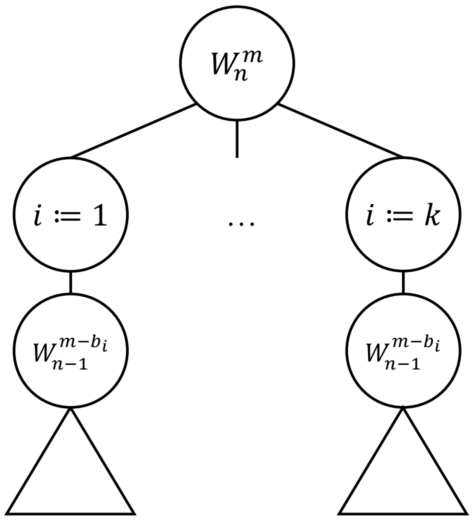

Figure 2: An AND/OR tree structure for W_n^m

Theorem 2. *There is a bijection between the set of all variants of the AND/OR tree for W_n^m and the set of simple directed lattice paths beginning at $(0,0)$, ending at (n,m) and consisting of steps $\{(1,b_1),(1,b_2),\ldots,(1,b_k)\}$.*

This bijection is based on the following idea: each selected child of the OR node labeled W_n^m determines the addition of the step s_i to the simple directed lattice path obtained by the subtree of the node labeled $W_{n-1}^{m-b_i}$.

Application of the method based on AND/OR trees to the obtained AND/OR tree structure for W_n^m allows us to develop new combinatorial generation algorithms (ranking, unranking and listing algorithms) for the set of simple directed lattice paths.

The following four types of directed lattice paths are distinguished [3]:

- walk: it is a directed lattice path that ends at (n,m) where m is not specified;
- bridge: it is a directed lattice path that ends at $(n,0)$;
- meander: it is a directed lattice path that ends at (n,m) where m is not specified and never falls below the x-axis;
- excursion: it is a directed lattice path that ends at $(n,0)$ and never falls below the x-axis.

Using Equation (1), we get the number of all simple walks and bridges:

- the number of all simple walks is $W_n = \sum\limits_{m=n\cdot\min b_i}^{n\cdot\max b_i} W_n^m$;
- the number of all simple bridges is $B_n = W_n^0$.

To prevent the simple directed lattice path from falling below the x-axis, we introduce the following recurrence with an additional initial condition:

Theorem 3. *The number of all simple directed lattice paths that begin at $(0,0)$, end at (n,m), consist of steps $\{(1,b_1),(1,b_2),\ldots,(1,b_k)\}$ and never fall below the x-axis* *can be calculated using the following recurrence:*

$$M_n^m = \sum_{i=1}^{k} M_{n-1}^{m-b_i}, \tag{2}$$

where $M_0^0 = 1$, $M_0^m = 0$ *for* $m \neq 0$ *and* $M_n^m = 0$ *for* $m < 0$.

Using Equation (2), we get the number of all simple meanders and excursions:

- the number of all simple meanders is $M_n = \sum_{m=0}^{n \cdot \max b_i} M_n^m$;
- the number of all simple excursions is $E_n = M_n^0$.

All the obtained formulas satisfies the requirements of the method based on AND/OR trees. Hence, it is possible to construct the corresponding AND/OR tree structures for W_n, B_n, M_n^m, M_n and E_n, and develop new combinatorial generation algorithms.

Conclusion

The main result of this work is the obtained recurrences for enumerating several different types of simple directed lattice paths. The obtained recurrences satisfy the requirements of the method based on the use of AND/OR trees. Using the obtained recurrences, the corresponding AND/OR tree structures were constructed, bijection rules between their variants and the considered lattice paths were defined, and algorithms for combinatorial generation were developed.

Acknowledgments

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This paper is dedicated to Professor Yilmaz SIMSEK on the occasion of his 60th anniversary.

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Relationship between combinatorial sets, generating functions and combinatorial generation algorithms

Yuriy Shablya [*1] *and Maria Perminova* [2]

Generating functions are actively used in the study of combinatorial sets, because they help to solve the problem of their enumeration. This article studies the relationship between combinatorial sets, generating functions and combinatorial generation algorithms. Some basic operations on generating functions associated with combinatorial sets are considered, as a result of which more complex combinatorial structures are obtained. Using formulas for calculating the coefficients of generating functions and the method based on AND/OR trees, an approach to develop combinatorial generation algorithms for such more complex combinatorial structures is obtained.

2020 MSC: 68R05, 05A15, 05C05

KEYWORDS: Combinatorial set, Generating function, AND/OR tree, Combinatorial generation algorithm

Introduction

A combinatorial set is a finite set whose elements have some structure and there is an algorithm for constructing the elements of this set. Combinatorics studies such combinatorial sets, including the issues of their enumeration and generation [1, 2, 4].

Generating functions are actively used in the study of combinatorial sets, because they help to solve the problem of their enumeration [5]. In this case, the coefficients of a generating function form the sequence of values of the cardinality function of some combinatorial set.

In addition, the cardinality function of a combinatorial set is often necessary when developing combinatorial algorithms for generating elements of these sets. Thus, there is a relationship between the designated concepts. This paper is devoted to the study of the relationship between combinatorial sets, generating functions and combinatorial generation algorithms.

Main results

Let two combinatorial sets F_n and G_n be given.

For the combinatorial set F_n, it is known the following:

- n is the size of the objects belonging to the combinatorial set F_n;
- $|F_n|$ is the number of objects belonging to the combinatorial set F_n (cardinality function);

- $F(x)$ is the ordinary generating function of the sequence of cardinality function values:
$$F(x) = \sum_{n \geq 0} f_n x^n = \sum_{n \geq 0} |F_n| x^n;$$
- A_F is an algorithm for generating objects belonging to the combinatorial set F_n.

For the combinatorial set G_n, it is known the following:

- n is the size of the objects belonging to the combinatorial set G_n;
- $|G_n|$ is the number of objects belonging to the combinatorial set G_n (cardinality function);
- $G(x)$ is the ordinary generating function of the sequence of cardinality function values:
$$G(x) = \sum_{n \geq 0} g_n x^n = \sum_{n \geq 0} |G_n| x^n;$$
- A_G is an algorithm for generating objects belonging to the combinatorial set G_n.

If we consider a combinatorial set H_n where $H(x) = F(x) + G(x)$, then:

- n is also the size of the objects belonging to the combinatorial set H_n;
- the following is true for the generating function $H(x)$:
$$H(x) = F(x) + G(x) = \sum_{n \geq 0} h_n x^n = \sum_{n \geq 0} (f_n + g_n) x^n.$$
  Hence:
$$h_n = f_n + g_n.$$
  This means that the objects of the combinatorial set H_n are obtained by the disjoint union of sets F_n and G_n, i.e. $H_n = F_n \sqcup G_n$.
- if we apply the method for developing combinatorial generation algorithms that is based on the use of AND/OR trees [3], then we construct the corresponding AND/OR tree structure for h_n (Figure 1). In this case, an algorithm A_H for generating objects belonging to the combinatorial set H_n is performed by generating variants of the obtained tree. Here, the children of nodes f_n and g_n are formed using combinatorial generation algorithms A_F and A_G.

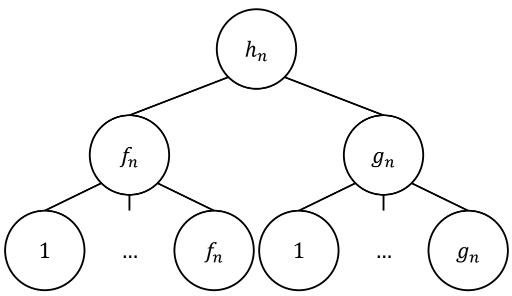

Figure 1: An AND/OR tree structure for h_n (when $H(x) = F(x) + G(x)$)

If we consider a combinatorial set H_n where $H(x) = F(x) - G(x)$, then:

- n is also the size of the objects belonging to the combinatorial set H_n;
- the following is true for the generating function $H(x)$:

$$H(x) = F(x) - G(x) = \sum_{n \geq 0} h_n x^n = \sum_{n \geq 0} (f_n - g_n) x^n.$$

  Hence:

$$h_n = f_n - g_n.$$

  This means that the objects of the combinatorial set H_n are obtained by the set difference of sets F_n and G_n, i.e. $H_n = F_n \setminus G_n$ where $G_n \subset F_n$.

- an algorithm A_H for generating objects belonging to the combinatorial set H_n is performed by generating objects belonging to the combinatorial set F_n (using combinatorial generation algorithms A_F) with skipping objects belonging to the combinatorial set G_n.

If we consider a combinatorial set H_n where $H(x) = F(x) \cdot G(x)$, then:

- n is also the size of the objects belonging to the combinatorial set H_n;
- the following is true for the generating function $H(x)$:

$$H(x) = F(x) \cdot G(x) = \sum_{n \geq 0} h_n x^n = \sum_{n \geq 0} \left(\sum_{i=0}^{n} f_i \cdot g_{n-i} \right) x^n.$$

  Hence:

$$h_n = \sum_{i=0}^{n} f_i \cdot g_{n-i}.$$

  This means that the objects of the combinatorial set H_n are obtained by the Cartesian product of sets F_i and G_{n-i} for all possible values of i, i.e. $H_n = (F_0 \times G_n) \cup (F_1 \times G_{n-1}) \cup \ldots \cup (F_{n-1} \times G_1) \cup (F_n \times G_0)$.

- if we apply the method for developing combinatorial generation algorithms that is based on the use of AND/OR trees [3], then we construct the corresponding AND/OR tree structure for h_n (Figure 2). In this case, an algorithm A_H for generating objects belonging to the combinatorial set H_n is performed by generating variants of the obtained tree. Here, the children of nodes f_n and g_n are formed using combinatorial generation algorithms A_F and A_G.

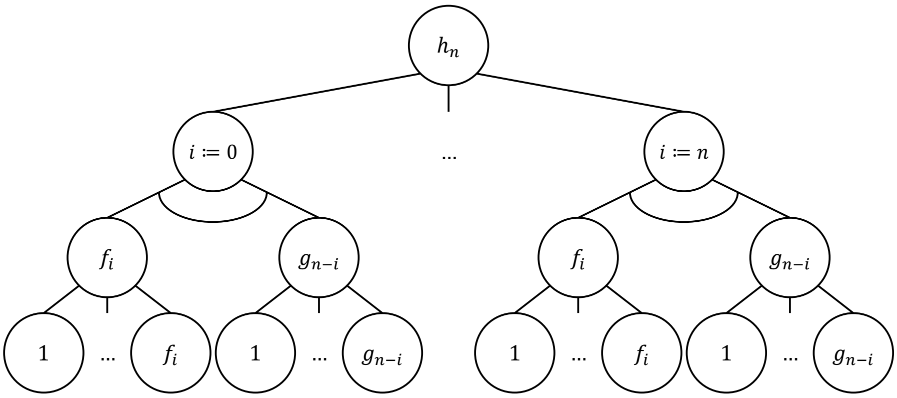

Figure 2: An AND/OR tree structure for h_n (when $H(x) = F(x) \cdot G(x)$)

Conclusion

The main result of this work is the obtained relationship between combinatorial sets, generating functions and combinatorial generation algorithms. Some basic operations (such as addition, subtraction and multiplication) on generating functions associated with combinatorial sets are considered. The result of these operations on generating functions is the generating function associated with more complex combinatorial structures. Using formulas for calculating the coefficients of generating functions and the method based on AND/OR trees, an approach to develop combinatorial generation algorithms for such more complex combinatorial structures is obtained.

Acknowledgments

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Second-order numerical method for solving singularly perturbed Volterra integro-differential equation

Zelal Temel

In this paper, we take into account the Volterra integro-differential equation with a singular perturbation. The major goal is to set up and examine a numerical strategy with uniform convergence in accordance with. The considered problem's numerical solution is discretized on a non-uniform mesh using the composite right-side rectangle rule for an integral component and implicit difference rules for the differential part. In order to demonstrate the effectiveness and viability of the suggested approach, numerical experiments are studied in the final stage.

2020 MSC: 65N12, 65N30, 65N06

KEYWORDS: Singularly perturbed, Uniform convergence, Numerical solution

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